(4)(a) Let $X = \mathbb{R}$ with the standard metric.

To show: (aa) $\mathbb{R}$ is complete
(ab) $\mathbb{R}$ is not bounded.

(ab) To show: There does not exist $M \in \mathbb{R}_{>0}$ such that if $x, y \in \mathbb{R}$ then $d(x,y) < M$.

Assume $M \in \mathbb{R}_{>0}$

Let $x = 0$ and $y = M+1$. Then

$$d(x,y) = |M+1-0| = M+1 \text{ and so } d(x,y) \not< M.$$

So there does not exist $M \in \mathbb{R}_{>0}$ such that if $x, y \in \mathbb{R}$ then $d(x,y) < M$.

(b) To show: $\mathbb{R}$ is not cover compact.

Let $\mathcal{S} = \{ B_1(x) \mid x \in \mathbb{Z} \} = \{ (x-1, x+1) \mid x \in \mathbb{Z} \}$

Then $\mathcal{S}$ is an open cover of $\mathbb{R}$.

To show: There does not exist a finite subcover of $\mathbb{R}$.

Proof by contradiction.

Assume $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in \mathbb{Z}$ such that

$$\mathbb{R} = B_1(x_1) \cup B_1(x_2) \cup \cdots \cup B_1(x_\ell).$$

~~For con~~ Let $M = \max\{x_1, x_2, \ldots, x_\ell\}$.

Let $y = M+2$. Since $x_1+1 < y, x_2+1 < y, \ldots, x_\ell+1 < y$ then

~~Then~~ $y \notin (x_1-1, x_1+1) \cup \cdots \cup (x_\ell-1, x_\ell+1)$

So $\{B_1(x_1), \ldots, B_1(x_\ell)\}$ is not a cover of y.

So $\mathcal{S}$ is an open cover of $\mathbb{R}$ with no finite subcover.

So $\mathbb{R}$ is not cover compact.

(c) To show: $\mathbb{R}$ is not ball compact.

Let $\varepsilon \in \mathbb{R}_{>0}$

To show: There do not exist a finite number of $B_\varepsilon(x)$ which cover $\mathbb{R}$.

Let $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in \mathbb{R}$.

Let $y = \max\{x_1, \ldots, x_\ell\} + 2\varepsilon$.

Since $x_1 + \varepsilon < y$, $x_2 + \varepsilon < y, \ldots, x_\ell + \varepsilon < y$ then

$y \notin (x_1 - \varepsilon, x_1 + \varepsilon) \cup \cdots \cup (x_\ell - \varepsilon, x_\ell + \varepsilon) = B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell)$.

So $\{B_\varepsilon(x) \mid x \in \mathbb{R}\}$ does not have a finite subcover.

So $\mathbb{R}$ is not ball compact.

(d) To show: $(0,1)$ is ball compact.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exist $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in (0,1)$ such that $(0,1) \subseteq B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell)$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $\ell \in \mathbb{Z}_{>0}$ such that $\frac{1}{\ell} < \frac{\varepsilon}{2}$.

Let $x_j = j\frac{\varepsilon}{2}$ for $j \in \{1, 2, \ldots, \ell\}$.

Then $B_\varepsilon(x_j) = (x_j - \varepsilon, x_j + \varepsilon)$ contains x_{j-1} and x_{j+1}

and $B_\varepsilon(x_1) = (x_1 - \varepsilon, x_1 + \varepsilon)$ contains 0

and $B_\varepsilon(x_\ell) = (x_\ell - \varepsilon, x_\ell + \varepsilon)$ with $x_\ell + \varepsilon > 1$.

So $(0,1) \subseteq (x_1 - \varepsilon, x_1 + \varepsilon) \cup (x_2 - \varepsilon, x_2 + \varepsilon) \cup \cdots \cup (x_\ell - \varepsilon, x_\ell + \varepsilon)$
$= B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell)$.

So $(0,1)$ is ball compact.

(d') To show: $(0,1)$ is not closed in $\mathbb{R}$.

If $\varepsilon \in \mathbb{R}_{>0}$ then $B_\varepsilon(0) \cap (0,1) \neq \emptyset$

If $\varepsilon \in \mathbb{R}_{>0}$ then $B_\varepsilon(1) \cap (0,1) \neq \emptyset$.

So 0 and 1 are close points to $(0,1)$.

Since $0 \notin (0,1)$ and $1 \notin (0,1)$, then $(0,1)$ is not closed in $\mathbb{R}$.

(e) To show: $(0,1)$ is not cover compact.

To show: There exists an open cover of $(0,1)$ which has no finite subcover

For $x \in (0,1)$ let

$$\varepsilon_x = \begin{cases} \frac{1}{2}|1-x|, & \text{for } x \geq \frac{1}{2} \\ \frac{1}{2}|x-0|, & \text{for } x < \frac{1}{2}. \end{cases}$$

and set

$$\mathcal{S} = \{ B_{\varepsilon_x}(x) \mid x \in (0,1) \}.$$

Then $\mathcal{S}$ is an open cover of $(0,1)$.

To show: $\mathcal{S}$ does not have a finite subcover.

To show: If $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in (0,1)$ then $B_{\varepsilon_{x_1}}(x_1) \cup \cdots \cup B_{\varepsilon_{x_\ell}}(x_\ell)$ is not ~~finite~~ a ~~sub~~cover of $(0,1)$.

Let $m = \min\{ x_1 - \varepsilon_{x_1}, x_2 - \varepsilon_{x_2}, \ldots, x_\ell - \varepsilon_{x_\ell} \}$ and

$M = \max\{ x_1 + \varepsilon_{x_1}, x_2 + \varepsilon_{x_2}, \ldots, x_\ell + \varepsilon_{x_\ell} \}$.

Then $m > 0$, by definition of ε_x, and

$M < 1$, by definition of ε_x, and

$B_{\varepsilon_{x_1}}(x_1) \cup \cdots \cup B_{\varepsilon_{x_\ell}}(x_\ell) \subseteq (m, M)$.

Since $\frac{m}{2} \in (0,1)$ and $\frac{m}{2} \notin (m, M)$ then

$\frac{m}{2} \in (0,1)$ and $\frac{m}{2} \notin B_{\varepsilon_{x_1}}(x_1) \cup \cdots \cup B_{\varepsilon_{x_\ell}}(x_\ell)$

So $\{B_{\varepsilon_1}(x_1), \ldots, B_{\varepsilon_\ell}(x_\ell)\}$ is not a cover of $(0,1)$.

So S has no finite subcover.

So $(0,1)$ is not cover compact.

(f) To show: $(0,1)$ is not Cauchy compact.

To show: There exists a Cauchy sequence $(a_1, a_2, \ldots)$ in $(0,1)$ which does not have a cluster point in $(0,1)$.

Let
$$(a_1, a_2, \ldots) = (\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots)$$

To show: (fa) $(a_1, a_2, \ldots)$ is Cauchy

(fb) $(a_1, a_2, \ldots)$ does not have a cluster point in $(0,1)$.

(fa) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $d(a_r, a_s) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $d(a_r, a_s) < \varepsilon$.

Let $N \in \mathbb{Z}_{>0}$ be such that $\frac{1}{N} < \varepsilon$.

To show: If $r, s \in \mathbb{Z}_{\geq N}$ then $d(a_r, a_s) < \varepsilon$.

Assume $r, s \in \mathbb{Z}_{\geq N}$ with $r < s$

To show: $d(a_r, a_s) < \varepsilon$.

$$d(a_r, a_s) = d(\tfrac{1}{r}, \tfrac{1}{s}) = \left|\tfrac{1}{r} - \tfrac{1}{s}\right| < \tfrac{1}{r} \leq \tfrac{1}{N} < \varepsilon.$$

So $(a_1, a_2, \ldots)$ is Cauchy.

(fb) To show: $(a_1, a_2, \ldots)$ does not have cluster point.

Let $x \in (0,1)$.

Let $N \in \mathbb{Z}_{>0}$ such that $\frac{1}{N+1} < x \leq \frac{1}{N}$.

Then $B_{\frac{1}{2N}}(x)$ contains at most one point of $(a_1, a_2, \ldots)$

So x is not a cluster point of $(a_1, a_2, \ldots)$.

So $(a_1, a_2, \ldots)$ does not have a cluster point in $(0,1)$.

So $(0,1)$ is not Cauchy compact.

(h) To show: $(0,1)$ is closed in $(0,1)$.

Since $(0,1)^c = \emptyset$ and $\emptyset$ is open in $(0,1)$ then $(0,1)$ is closed in $(0,1)$.

(g) Let $Y = \mathbb{R}$ with metric $\rho(x,y) = \min\{|x-y|, 1\}$. Let $A = Y$.

To show: (ga) A is bounded

(gb) A is not ball compact.

(ga) To show: There exists $M \in \mathbb{R}_{>0}$ such that if $x, y \in A$ then $\rho(x,y) < M$.

Let $M = 2$.

To show: If $x, y \in A$ then $\rho(x,y) < M$.

Assume $x, y \in A$

To show: $\rho(x,y) < M$.

$\rho(x,y) = \min\{|x-y|, 1\} \leq 1 < 2 = M$.

So A is bounded.

(9b) To show: A is not ball compact.

To show: There exists $\varepsilon\in\mathbb{R}_{>0}$ such that there does not exist $\ell\in\mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell\in\mathbb{R}$ with $B_\varepsilon(x_1)\cup\cdots\cup B_\varepsilon(x_\ell)=\mathbb{R}$.

Let $\varepsilon=\frac{1}{2}$.

To show: There does not exist $\ell\in\mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell\in\mathbb{R}$ with $B_\varepsilon(x_1)\cup\cdots\cup B_\varepsilon(x_\ell)=\mathbb{R}$.

Let $\ell\in\mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell\in\mathbb{R}$.

To show: $B_\varepsilon(x_1)\cup\cdots\cup B_\varepsilon(x_\ell)\neq\mathbb{R}$.

To show: There exists $z\in\mathbb{R}$ such that $z\notin B_\varepsilon(x_1)\cup\cdots\cup B_\varepsilon(x_\ell)$.

Let $z=\max\{x_1,\ldots,x_\ell\}+2$.

Then

$$B_\varepsilon(x_1)\cup\cdots\cup B_\varepsilon(x_\ell)=B_{\frac{1}{2}}(x_1)\cup\cdots\cup B_{\frac{1}{2}}(x_\ell)$$
$$=(x_1-\tfrac{1}{2},x_1+\tfrac{1}{2})\cup\cdots\cup(x_\ell-\tfrac{1}{2},x_\ell+\tfrac{1}{2})$$

Since $z>x_1+\frac{1}{2}$, $z>x_2+\frac{1}{2}$, $\ldots$, $z>x_\ell+\frac{1}{2}$ then $z\notin B_\varepsilon(x_1)\cup\cdots\cup B_\varepsilon(x_\ell)$.

So $B_\varepsilon(x_1)\cup\cdots\cup B_\varepsilon(x_\ell)\neq\mathbb{R}$

So $A=\mathbb{R}$ is not ball compact.