(2)(a) Let $(H, \langle\ \rangle)$ be an infinite dimensional Hilbert space which is separable.

Let $b_1', b_2', \ldots$ be a countable dense set.

Let $\{b_1, b_2, \ldots\}$ be a subset of $\{b_1', b_2', \ldots\}$ which consists of linearly independent elements.

Use $\{b_1, b_2, \ldots\}$ to produce an orthonormal sequence $(a_1, a_2, \ldots)$ by using Gram-Schmidt,

$$a_1 = b_1, \quad a_{n+1} = \frac{b_{n+1} - \langle b_{n+1}, a_1\rangle a_1 - \cdots - \langle b_{n+1}, a_n\rangle a_n}{\| b_{n+1} - \langle b_{n+1}, a_1\rangle a_1 - \cdots - \langle b_{n+1}, a_n\rangle a_n \|}.$$

Let $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, $e_3 = (0, 0, 1, 0, \ldots)$ and define a linear transformation

$$\Phi : H \longrightarrow \ell^2 \text{ by } \Phi(a_i) = e_i.$$

To show: (a) Φ is a function

(b) If $x, y \in H$ then $\langle \Phi(x), \Phi(y)\rangle = \langle x, y\rangle$

(c) Φ is bijective.

Let $B' = \{b_1', b_2', \ldots\}$ and $B = \{b_1, b_2, \ldots\}$.

Then $B \subseteq B' \subseteq \mathrm{span}(B)$.

So ~~span(B)~~ $\overline{B'} \subseteq \overline{\mathrm{span}(B)}$ and so $\overline{\mathrm{span}(B)} = H$.

Thus, if $x \in X$ then

$$x = \sum_{n=1}^{\infty} \langle x, a_n \rangle a_n$$

Thus $\Phi(x) = (\langle x, a_1\rangle, \langle x, a_2\rangle, \ldots)$.

(b) Assume

~~Let~~ $x, y \in H$. ~~and~~

To show: $\langle \Phi(x), \Phi(y) \rangle = \langle x, y \rangle$.

$$\langle x, y \rangle = \left\langle \sum_{n=1}^{\infty} \langle x, a_n\rangle a_n, \sum_{m=1}^{\infty} \langle y, a_m \rangle a_m \right\rangle$$

$$= \sum_{n=1}^{\infty} \langle x, a_n \rangle \langle y, a_n \rangle$$

$$= \langle (\langle x, a_1\rangle, \langle x, a_2\rangle, \ldots), (\langle y, a_1\rangle, \langle y, a_2\rangle, \ldots) \rangle$$

$$= \langle \Phi(x), \Phi(y) \rangle$$

(c) To show (ca) Φ is injective

(cb) Φ is surjective

(ca) Since $\langle \Phi(x), \Phi(x) \rangle = \langle x, x \rangle$ ~~if~~ then

~~$\|\Phi(x)$~~ $\|\Phi(x)\|_2 = \|x\|$

Since $d(x, y) = d(\Phi(x), \Phi(y))$ then Φ is an isometry.

So Φ is injective.

(2)(b) To show: (ba) ℓ^∞ is a metric space.

(bb) ℓ^∞ is not separable.

(bb) ~~Let~~ For $Q \subseteq \mathbb{Z}_{>0}$ ~~let~~ with $Q \neq \emptyset$ let

$$e_Q = (a_1, a_2, \ldots) \text{ with } a_i = \begin{cases} 1, & \text{if } i \in Q, \\ 0, & \text{if } i \notin Q. \end{cases}$$

Then
$$\|e_Q\|_\infty = \sup\{|a_1|, |a_2|, \ldots\} = 1$$

and if $P, Q \subseteq \mathbb{Z}_{>0}$ and $P \neq Q$ then

$$d(e_P, e_Q) = \|e_P - e_Q\|_\infty = \sup\{|1-1|, |1-0|, |0-1|, |0-0|\}$$
$$= \sup\{0, 1, 1, 0\} = 1.$$

Then $\{e_Q \mid Q \subseteq \mathbb{Z}_{>0}\}$ is uncountable.

To show: ℓ^∞ does not contain a countable dense set.

To show: If $C = (c_1, c_2 \ldots)$ is a countable subset of ℓ^∞ then C is not dense in ℓ^∞.

~~If~~ If C is dense in ℓ^∞ then

$\{B_{\frac{1}{2}}(c_j) \mid j \in \mathbb{Z}_{>0}\}$ is a cover of ℓ^∞.

So there exists $B_{\frac{1}{2}}(c_j)$ and $P, Q \in \mathbb{Z}_{>0}$ with $P \neq Q$ and $e_P, e_Q \in B_{\frac{1}{2}}(c_j)$.

Then $1 = d(e_P, e_Q) \leq d(e_P, c_j) + d(e_Q, c_j) < \frac{1}{2} + \frac{1}{2} = 1.$

To show: $\| \ \| : \ell^\infty \to \mathbb{R}_{\geq 0}$ is a norm.

To show: (baa) If $x \in \ell^\infty$ and $c \in \mathbb{R}$ then $\|cx\| = |c|\,\|x\|$.

(bab) If $x \in \ell^\infty$ and $\|x\| = 0$ then $x = 0$.

(bac) If $x, y \in \ell^\infty$ then $\|x+y\| \leq \|x\| + \|y\|$.

(a) Assume $x = (x_1, x_2, \ldots)$ and $c \in \mathbb{R}$.

Then $cx = (cx_1, cx_2, \ldots)$ and

$$\begin{aligned}
\|cx\|_\infty &= \sup\{ |cx_1|, |cx_2|, \ldots \} \\
&= \sup\{ |c||x_1|, |c||x_2|, \ldots \} \\
&= |c| \sup\{ |x_1|, |x_2|, \ldots \} \\
&= |c|\, \|x\|_\infty
\end{aligned}$$

(b) Assume $x = (x_1, x_2, \ldots)$ and $\|x\|_\infty$.

Then $\sup\{ |x_1|, |x_2|, \ldots \} = 0$.

So $|x_1| = 0, |x_2| = 0, \ldots$. So $x_1 = 0, x_2 = 0, \ldots$

So $x = 0$.

(c) Assume $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$

To show: $\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$.

$$\begin{aligned}
\|x+y\|_\infty &= \sup\{ |x_1+y_1|, |x_2+y_2|, \ldots \} \\
&\leq \sup\{ |x_1|+|y_1|, |x_2|+|y_2|, \ldots \} \\
&\leq \sup\{ |x_1| + \|y\|_\infty, |x_2| + \|y\|_\infty, \ldots \} \\
&\leq \sup\{ |x_1|, |x_2|, \ldots \} + \|y\|_\infty = \|x\|_\infty + \|y\|_\infty.
\end{aligned}$$

(3) Let $a \in \mathbb{R}_{>0}$ and

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \quad \text{for } x \in \mathbb{R}_{>0}.$$

(a) To show: If $x \in \mathbb{R}_{>0}$ then $f(x) \geq \sqrt{a}$.

Assume $x \in \mathbb{R}_{>0}$

To show: $f(x) \geq \sqrt{a}$.

$$\cancel{f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) = \frac{1}{2}\left(\frac{x^2 + a}{x}\right) = \frac{1}{4}(}$$

To show: $\frac{1}{2}\left(x + \frac{a}{x}\right) \geq \sqrt{a}$.

To show: $\frac{1}{4}\left(x^2 + 2a + \frac{a^2}{x^2}\right) \geq a$.

To show: $\frac{1}{4}\left(x^2 - 2a + \frac{a^2}{x^2}\right) \geq 0$.

To show: $\frac{1}{4}\left(x - \frac{a}{x}\right)^2 \geq 0$.

Yes this is true for $x \in \mathbb{R}_{>0}$.

(b) To show: f is a contraction mapping.

To show: There exists $\alpha \in (0,1)$ such that

if $x, y \in [\sqrt{a}, \infty)$ then $d(f(x), f(y)) \leq \alpha\, d(x,y)$

Let $\alpha = \frac{1}{2}$

To show: $d(f(x), d(f(y)) \leq \alpha d(x,y)$.

$$d(f(x), f(y)) = d\left(\tfrac{1}{2}(x+\tfrac{a}{x}), \tfrac{1}{2}(y+\tfrac{a}{y})\right)$$

$$= \left|\tfrac{1}{2}(y+\tfrac{a}{y}) - \tfrac{1}{2}(x+\tfrac{a}{x})\right| = \tfrac{1}{2}\left|(y-x) + \frac{a(x-y)}{xy}\right|$$

$$= \tfrac{1}{2}\left(1-\frac{a}{xy}\right)|y-x| \leq \tfrac{1}{2}|y-x| = \tfrac{1}{2}d(x,y),$$

(since $x,y \geq \sqrt{a}$ then $\frac{a}{xy} \leq 1$ and $1-\frac{a}{xy} \leq 1$).

So f is a contraction mapping.

(c) Let $x_0 \in \mathbb{R}_{>0}$ with $x_0 > \sqrt{a}$. Let

$$x_{n+1} = f(x_n), \text{ for } n \in \mathbb{Z}_{\geq 0}.$$

Since f is contractive the Banach fixed point theorem gives that $z = \lim\limits_{n\to\infty} x_n$ exists and is the unique fixed point of f.

So $f(z) = z$

So $\frac{1}{2}(z+\frac{a}{z}) = z$

Thus $\frac{1}{2}\frac{a}{z} = \frac{1}{2}z$ and $a = z^2$ so that $z = \sqrt{a}$.

So $\lim\limits_{n\to\infty} x_n = \sqrt{a}$.

This solution will receive at least 8/10 marks. A solution with some further justification of why $z = \lim\limits_{n\to\infty} x_n$ exists and is the fixed point of f will receive more marks.