Solutions, 23 Oct. 2015

(1) Compute $\|W\|$ and an eigenvector of $W$ with eigenvalue $\|W\|$.

The matrix of $W: \mathbb{C}^5 \to \mathbb{C}^5$ in the standard basis is

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 6 & 7 & 8 & 9 \\ 3 & 7 & 10 & 11 & 12 \\ 4 & 8 & 11 & 13 & 14 \\ 5 & 9 & 12 & 14 & 15 \end{pmatrix}.$$

Write $\|W\| = \|B\|$ for convienience

Then
$$\|B\| = \sup\left\{ \frac{\|Bu\|}{\|u\|} \;\middle|\; u \in \mathbb{C}^5 \right\} = \{ \|Bu\| \mid \|u\| = 1 \}$$

and, since $B = \overline{B}^t$, $B$ is self adjoint and

$$\|B\| = \sup\{ |\langle Bu, u\rangle| \mid \|u\| \le 1 \}$$
$$= \sup\left\{ \frac{|\langle Bu, u\rangle|}{\|u\|^2} \;\middle|\; u \in \mathbb{C}^5 \right\}.$$

If $u_1 = (1,1,1,1,1)$ then $\|u_1\| = \sqrt{1^2+1^2+1^2+1^2+1^2} = \sqrt{5}$

and

$$Bu_1 = (15, 32, 43, 50, 55) \text{ so that}$$

$$\langle Bu_1, u_1\rangle = 47 + 43 + 105 = 195 \text{ and } \frac{\langle Bu_1, u_1\rangle}{\sqrt{5}} = \frac{195}{5} = 39$$

So $\|W\| = \|B\| \ge 39$

Note that $\frac{1}{15} Bu_1 \approx (1, 2, 3, 3, 3)$

Let $u_2 = (1, 2, 3, 3, 3)$ and $\|u_2\|^2 = 1^2 + 4 + 9 + 9 + 9 = 32$

Then

$$Bu_2 = (1+4+9+12+15,\ 2+12+21+24+27,\ 3+14+30+33+36,$$
$$4+16+33+39+42,\ 5+18+36+42+45)$$
$$= (41, 85, 116, 134, 146)$$

and $\dfrac{\langle Bu_2, u_2\rangle}{\|u_2\|^2} = \dfrac{41+170+348+402+438}{32}$

$$= \frac{1399}{32} = 40 + \frac{119}{32} = 40+3+\frac{23}{32} \approx 43.6.$$

So $\|B\| = \|W\| \geq 43.6.$

~~Since~~ $B$ is self adjoint there is a matrix $K$ with $KK^* = 1$ and

$$KBK^{-1} = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ 0 & & & & \lambda_5 \end{pmatrix}$$ and if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$ then $|\lambda_1| = \|B\| = \|KBK^{-1}\|.$

Then

$$\frac{1}{\|B\|} KBK^{-1} = \frac{1}{\lambda_1} KBK^{-1} = \begin{pmatrix} \lambda_1/\lambda_1 & & & & 0 \\ & \lambda_2/\lambda_1 & & & \\ & & \lambda_3/\lambda_1 & & \\ & & & \lambda_4/\lambda_1 & \\ 0 & & & & \lambda_5/\lambda_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & 0 \\ & a_2 & & & \\ & & a_3 & & \\ & & & a_4 & \\ 0 & & & & a_5 \end{pmatrix}, \text{ with } a_2 < 1, a_3 < 1, a_4 < 1, a_5 < 1.$$

So $KB^2K^{-1} = \begin{pmatrix} 1 & & & & \\ & a_2^2 & & & \\ & & a_3^2 & & \\ & & & a_4^2 & \\ & & & & a_5^2 \end{pmatrix}$ and $KB^3K^{-1} = \begin{pmatrix} 1 & & & & 0 \\ & a_2^3 & & & \\ & & a_3^3 & & \\ & & & a_4^3 & \\ 0 & & & & a_5^3 \end{pmatrix}$

and $\lim_{k\to\infty} KB^kK^{-1} = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}.$

So, if $u \in \mathbb{C}^5$, then

$u_1 = Bu, u_2 = B^2u, u_3 = B^3u, \ldots$

is a sequence of vectors and

$\lim_{k\to\infty} u_k = \lim_{k\to\infty} B^k u$ is an eigenvector of $B$.

So $u_1 = Bu, u_2 = B^2u, \ldots$

In our case, with $u_1 = (1,1,1,1,1)$ we have

$u_2 = \frac{Bu_1}{15} \approx (1,2,3,3,3)$ and

$B u_2 \approx (41, 85, 116, 134, 146)$

So we expect ~~the big~~ $\|B\| \approx 44$

and the eigenvector to be approximately equal to $(41, 85, 116, 134, 146)$.

In fact, Wolfram alpha, with

eigenvalues{{1,2,3,4,5},{2,6,7,8,9},{3,7,10,11,12},{4,8,11,13,14},{6,9,12,14,15}}

produces eigenvalues

$\lambda_1 \approx 44.2126$, $\lambda_3 \approx 1.26641$, $\lambda_5 \approx 0.241956$

$\lambda_2 \approx -1.43905$, $\lambda_4 \approx 0.718115$, and the eigenvector corresponding to the largest eigenvalue is

$$v_1 \approx (0.282093, 0.585915, 0.788872, 0.91288, 1)$$

Our computation (by hand)

$$v_1 \approx \frac{B^4 u_1}{146} = \left(\frac{41}{146}, \frac{85}{146}, \frac{116}{146}, \frac{134}{146}, 1\right)$$

$$\approx (0.2808, 0.58219, 0.7945, 0.9178, 1)$$

where the last line was done by calculator.

So, by hand we were able to get a very good approximation (after only two steps!).

We had estimated

$\|W\| \approx 44$ and Wolfram alpha estimates

$\|W\| \approx 44.2126$.

For computing the norm $\|T\|$

Let

$$C = \bar{A}^t A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 0 & 14 & 15 \\ 16 & 0 & 2 & 0 & 20 \\ 1 & 0 & 3 & 4 & 10 \end{pmatrix}^t \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 0 & 14 & 15 \\ 16 & 0 & 2 & 0 & 20 \\ 1 & 0 & 3 & 4 & 10 \end{pmatrix}$$

Then $C$ is selfadjoint and since, if $\bar{A}^t A u = \gamma u$ use

$$\|Au\|^2 = \langle Au, Au\rangle = \langle u, \bar{A}^t A u\rangle = \langle u, \gamma u\rangle = \bar{\gamma}\|u\|^2 = \gamma\|u\|^2$$

~~then~~ to show

$\|A\| = \sqrt{\gamma}$, where $\gamma$ is the largest eigenvalue of $C$.

As we did for computing the norm $\|W\|$

we know

$$\frac{\langle Cu, u\rangle}{\|u\|^2} \quad \text{with } u = (1,1,1,1,1) \text{ and } \|u\| = \sqrt{5}$$

is likely not a bad estimate of $\|C\|$.

So

$$\|A\| \approx \sqrt{\frac{\langle Cu, u\rangle}{\|u\|^2}} = \frac{\sqrt{\langle Au, Au\rangle}}{\sqrt{5}}$$

and

$$Au = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 0 & 20 & 20 & 20 \\ 1 & 0 & 3 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 40 \\ 52 \\ 38 \\ 18 \end{pmatrix}$$

and $\langle Au, Au\rangle = 15^2 + 40^2 + 52^2 + 38^2 + 18^2$

so

$$\frac{\sqrt{\langle Au, Au\rangle}}{\|u\|} = \cancel{\frac{1}{\sqrt{5}}\sqrt{15^2+40^2}} \sqrt{\frac{15^2 + 40^2 + 52^2 + 38^2 + 18^2}{5}}$$

$$\approx \sqrt{5\cdot 3 + 8\cdot 40 + 10\cdot 52 + 7\cdot 38 + 3\cdot 18}$$

$$\approx \sqrt{15 + 320 + 520 + 280 + 60}$$

$$\approx \sqrt{1200} \approx 10\cdot\sqrt{12} \approx 10\cdot 3.7 \approx 37.$$

In fact Wolfram alpha says

$$C = \begin{pmatrix} 415 & 176 & 86 & 216 & 560 \\ 176 & 197 & 62 & 239 & 260 \\ 86 & 62 & 86 & 96 & 165 \\ 216 & 239 & 96 & 309 & 360 \\ 560 & 260 & 165 & 360 & 850 \end{pmatrix}$$

with largest eigenvalue $1537.87$

and $\sqrt{1537.87} \approx 39.216$.

So our estimate of 37 was not too bad.

(1b) Let $P$ be the matrix of $T^*$.

Let $e_1, e_2, e_3, e_4, e_5$ be the standard basis of $\mathbb{C}^5$.

Then

$$Te_i = \sum_{j=1}^{5} \langle Te_i, e_j \rangle e_j = \sum_{j=1}^{5} \left\langle \sum_{k=1}^{5} a_{ki} e_k, e_j \right\rangle e_j$$

$$= \sum_{j=1}^{5} a_{ji} e_j .$$

Then

$$a_{ji} = \langle Te_i, e_j \rangle = \langle e_i, T^* e_j \rangle = \overline{\langle T^* e_j, e_i \rangle}$$

So $\langle T^* e_i, e_j \rangle = \bar{a}_{ij}$. So $P = \bar{A}^t$.

So the matrix of $T^*$ is

$$\bar{A}^t = \begin{pmatrix} 1 & 6 & 11 & 16 & 1 \\ 2 & 7 & 12 & 0 & 0 \\ 3 & 8 & 0 & 2 & 3 \\ 4 & 9 & 14 & 0 & 4 \\ 5 & 10 & 15 & 20 & 10 \end{pmatrix}$$

and the matrix of $W^*$ is

$$\bar{B}^t = B^t = B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 6 & 7 & 8 & 9 \\ 3 & 7 & 10 & 11 & 12 \\ 4 & 8 & 11 & 13 & 14 \\ 5 & 9 & 12 & 14 & 15 \end{pmatrix}$$

(1c) A <u>compact operator</u> is $T: V \to W$ such that

$$\overline{\{Tx \mid \|x\|=1\}} \text{ is compact in } W.$$

To show: $T: V \to V$ is compact, where $V = \mathbb{C}^5$.

To show: (a) $\overline{\{Tx \mid \|x\|=1\}} \subseteq \mathbb{C}^5$ is closed.

(b) $\overline{\{Tx \mid \|x\|=1\}} \subseteq \mathbb{C}^5$ is bounded.

(a) $\overline{\{Tx \mid \|x\|=1\}}$ is closed since the closure of $\{Tx \mid \|x\|=1\}$ is the smallest closed set containing $\{Tx \mid \|x\|=1\}$.

(b) To show: $\overline{\{Tx \mid \|x\|=1\}}$ is bounded.

If $x \in \mathbb{C}^5$ with $x = (x_1, x_2, x_3, x_4, x_5)$ then

$$Tx = T\left(\sum_{i=1}^{5} x_i e_i\right) = \sum_{i=1}^{5} x_i Te_i = \sum_{i=1}^{5} x_i \sum_{j=1}^{5} a_{ji} e_j$$

$$= \sum_{i,j=1}^{5} a_{ji} x_i e_j.$$

Let $a_{max} = \sup\{a_{ji} \mid i,j \in \{1,2,3,4,5\}\}$

In our case $a_{max} = 20$ since the largest entry in the matrix $A$ is 20.

Assume $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5$ and $\|x\| = 1$. Then

$$1 = \|x\|^2 = \langle x, x \rangle = \sum_{i=1}^{5} |x_i|^2. \quad \text{So } |x_i| \le 1.$$

Then

$$\|Tx\|^2 = |\langle Tx, Tx \rangle| = \left| \left\langle \sum_{i,j} x_i a_{ji} e_j, \sum_{k,l} x_l a_{kl} e_k \right\rangle \right|$$

$$= \left| \sum_{i,j,k,l} x_i a_{ji} \bar{x}_l \bar{a}_{kl} \langle e_j, e_k \rangle \right|$$

$$= \left| \sum_{i,j,l} x_i \bar{x}_l a_{ji} \bar{a}_{jl} \right| \le \sum_{i,j,l} |x_i \bar{x}_l a_{ji} \bar{a}_{jl}|$$

$$= \sum_{i,j,l} |x_i| |\bar{x}_l| |a_{ji}| |\bar{a}_{ji}| \le \sum_{i,j,l} 1 \cdot 1 \cdot |a_{max}| \cdot |a_{max}|$$

$$= \sum_{i,j,l=1}^{5} |a_{max}|^2 = 5^3 |a_{max}|^2.$$

So $\|Tx\|$ is bounded by $5^3 |a_{max}|^2$.

So $\overline{\{Tx \mid \|x\| = 1\}}$ is bounded by $5^3 |a_{max}|^2$.

So $\overline{\{Tx \mid \|x\| = 1\}}$ is closed and bounded in $\mathbb{C}^5$.

So $T : \mathbb{C}^5 \to \mathbb{C}^5$ is a compact operator.

The same argument holds for the operator

$W : \mathbb{C}^5 \to \mathbb{C}^5$.

(1e) To construct an orthonormal basis of eigenvectors of $W$.

In the first part we found (approximately) an eigenvector

$v_1 \approx (41, 85, 116, 134, 146)$ of eigenvalue $\|W\| \approx 44$.

Let $e_2 = (0,1,0,0,0)$, $e_3 = (0,0,1,0,0)$ $e_4 = (0,0,0,1,0)$ $e_5 = (0,0,0,0,1)$ so that if $u_1 = \frac{v_1}{\|v_1\|}$ then $\{u_1, e_2, e_3, e_4, e_5\}$ is a basis of $\mathbb{C}^5$.

Use Gram-Schmidt to make an orthonormal basis $\{u_1, v_2, v_3, v_4, v_5\}$ with

$$v_2 = \frac{e_2 - \langle e_2, u_1\rangle u_1}{\|e_2 - \langle e_2, u_1\rangle u_1\|} = \frac{(0,1,0,0,0) - \frac{85}{\|v_1\|} v_1}{\|e_2 - \frac{85}{\|v_1\|} v_1\|}$$

$$v_3 = \frac{e_3 - \langle e_3, u_1\rangle u_1 - \langle e_3, v_2\rangle v_2}{\|e_3 - \langle e_3, u_1\rangle u_1 - \langle e_3, v_2\rangle v_2\|}$$

$$v_4 = \frac{e_4 - \langle e_4, u_1\rangle u_1 - \langle e_4, v_2\rangle v_2 - \langle e_4, v_3\rangle v_3}{\|e_4 - \langle e_4, u_1\rangle u_1 - \langle e_3, v_2\rangle v_2 - \langle e_3, v_3\rangle v_3\|}$$

$$v_5 = \frac{e_5 - \langle e_5, u_1\rangle u_1 - \langle e_5, v_2\rangle v_2 - \langle e_5, v_3\rangle v_3 - \langle e_5, v_4\rangle v_4}{\|e_5 - \langle e_5, u_1\rangle u_1 - \langle e_5, v_2\rangle v_2 - \langle e_5, v_3\rangle v_3 - \langle e_5, v_4\rangle v_4\|}.$$

Let $K_1$ be the change of basis matrix from the basis $\{u_1, e_2, e_3, e_4, e_5\}$ to $\{u_1, v_2, v_3, v_4, v_5\}$.

Then, the matrix of $W$ with respect to the new basis $\{u_1, v_2, v_3, v_4, v_5\}$, is

$$K_1 B K_1^{-1} = \begin{pmatrix} \|W\| & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & W_1 & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}$$

where $W_1$ is a $4\times 4$ matrix. Then $W_1$ is selfadjoint since

$$\left(\overline{K_1 B K_1^{-1}}\right)^t = \left(\overline{K_1 B \overline{K_1}^t}\right)^t = \left(\overline{K_1}^t\right)^t \overline{B}^t \overline{K_1}^t = K_1 \overline{B}^t K_1^{-1}$$
$$= K_1 B K_1^{-1}, \text{ so that } \overline{W_1}^t = W_1 .$$

Thus we can repeat what we did for $W$ with $W_1$ to find $K_2$ with $K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & \tilde{K}_2 & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}$ so that

$$K_2 K_1 B K_1^{-1} K_2^{-1} = \begin{pmatrix} \|W\| & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & \tilde{K}_2 W_1 \tilde{K}_2^{-1} & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix} = \begin{pmatrix} \|W\| & 0 & 0 & 0 & 0 \\ 0 & \|W_1\| & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & 0 & & W_2 & \\ 0 & 0 & & & \end{pmatrix}$$

Repeating the process 2 more times produces an orthonormal basis $\{u_1, u_2, u_3, u_4, u_5\}$ of eigenvectors of $W$ since

$$K_4 K_3 K_2 K_1 B K_1^{-1} K_2^{-1} K_3^{-1} K_4^{-1} = \begin{pmatrix} \|W\| & 0 & 0 & 0 & 0 \\ 0 & \|W_1\| & 0 & 0 & 0 \\ 0 & 0 & \|W_2\| & 0 & 0 \\ 0 & 0 & 0 & \|W_3\| & 0 \\ 0 & 0 & 0 & 0 & \|W_4\| \end{pmatrix}$$

(18) To show: $T$ has an eigenvector.

To show: There exists $\lambda$ such that
$$\{v \in \mathbb{C}^5 \mid Tv = \lambda v\} \neq 0.$$

To show: There exists $\lambda$ such that
$$\{v \in \mathbb{C}^5 \mid (T-\lambda)v = 0\} \neq 0.$$

To show: There exists $\lambda$ such that $\ker(T-\lambda) \neq 0$.

To show: There exists $\lambda$ such that $T-\lambda$ is not invertible.

To show: There exist $\lambda$ such that $\det(T-\lambda) = 0$.

To show: $\det(T-t)$ has a root (the charateristic polynomial has a root)..

Since $A$ has integer entries and is $5\times 5$
$$\det(A-t) = d_0 + d_1 t + d_2 t^2 + d_3 t^3 + d_4 t^4 - t^5$$
with $d_0, d_1, d_2, d_3, d_4 \in \mathbb{Z}$.

If $t \in \mathbb{R}_{>0}$ and $t$ is very large then $\det(A-t) < 0$.
If $t \in \mathbb{R}_{<0}$ and $t$ is very large then $\det(A-t) > 0$.

~~So then~~ Since $\det(A-t)$ is a polynomial, $\det(A-t)$ is a continuous function (from $\mathbb{R} \to \mathbb{R}$) and so there exists $\lambda \in \mathbb{R}$ with
$$\det(A-\lambda) = 0. //$$