<u>Tutorial</u>: 06 October 2015, Metric and Hilbert.

(1) Let H be a Hilbert space. Define

$\varphi : H \to H^*$ by

$$\begin{aligned} H &\longrightarrow H^* \\ x &\longmapsto \varphi_x : H \to \mathbb{R} \\ &\qquad y \longmapsto \langle x, y \rangle \end{aligned}$$

Show that if $x \in H$ then $\|x\| = \|\varphi_x\|$.

(2) Let H be a Hilbert space. Define

$\varphi : H \to H^*$ by

$$\begin{aligned} H &\longrightarrow H^* \\ x &\longmapsto \varphi_x : H \to \mathbb{R} \\ &\qquad y \longmapsto \langle x, y \rangle. \end{aligned}$$

Show that φ is injective.

(3) Let H be a Hilbert space. Define

$\varphi : H \to H^*$ by

$$\begin{aligned} H &\longrightarrow H^* \\ x &\longmapsto \varphi_x : H \to \mathbb{C} \\ &\qquad y \longmapsto \langle x, y \rangle. \end{aligned}$$

Show that φ is surjective.

(4) Let H be a Hilbert space. If H has a countable dense set C then there exists an <u>orthonormal sequence</u> $(e_1, e_2, \dots)$ in H with

$\overline{\operatorname{span}\{e_1, e_2, \dots\}} = H$.