Tutorial 6, Metric and Hilbert Spaces, 08.09.2015

Homeomorphisms and isometries

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

A <u>homeomorphism</u> from X to Y is a function $\varphi : X \to Y$ such that

(a) φ is continuous

(b) the inverse function $\varphi^{-1} : Y \to X$ exists

(c) $\varphi^{-1} : Y \to X$ is continuous.

Let (X, d_X) and (Y, d_Y) be metric spaces.

An <u>isometry</u> from X to Y is a function $\varphi : X \to Y$ such that

if $x_1, x_2 \in X$ then $d_Y(\varphi(x_1), \varphi(x_2)) = d(x_1, x_2)$

<u>HW</u>: Show that if $\varphi : X \to Y$ is an isometry then φ is injective.

<u>HW</u>: Show that the inclusion $\varphi : \mathbb{Q} \to \mathbb{R}$ is an isometry that is not surjective.

<u>HW</u>: Show that $\mathbb{R}$ with the cofinite topology is not Hausdorff and is compact.

Tutorial 6, 08.09.2015 Metric and Hilbert. (2)

HW: If $\varphi: X \to Y$ is an isometry then φ is injective.

Proof: Assume $\varphi: X \to Y$ is an isometry.

To show: φ is injective.

To show: If $x_1, x_2 \in X$ and $\varphi(x_1) = \varphi(x_2)$ then $x_1 = x_2$.

Assume $x_1, x_2 \in X$ and $\varphi(x_1) = \varphi(x_2)$.

To show: $x_1 = x_2$

To show: $d_X(x_1, x_2) = 0$.

$$d_X(x_1, x_2) = d_Y(\varphi(x_1), \varphi(x_2)) = 0$$

So $x_1 = x_2$

So φ is injective.

Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. Show that

$$\lim_{n\to\infty} x_n = z \text{ if and only if } \lim_{n\to\infty} d(x_n, z) = 0.$$

<u>Proof</u> $\Rightarrow$ Assume $\lim_{n\to\infty} x_n = z$.

To show: $\lim_{n\to\infty} d(x_n, z) = 0$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $|d(x_n, z)| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Since $\lim_{n\to\infty} x_n = z$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $x_n \in B_\varepsilon(z)$.

Let $\ell = N$.

To show: If $n \in \mathbb{Z}_{\geq N}$ then $|d(x_n, z)| < \varepsilon$.

Assume $n \in \mathbb{Z}_{\geq N}$.

To show: $d(x_n, z) < \varepsilon$.

Since $x_n \in B_\varepsilon(z)$ then $d(x_n, z) < \varepsilon$.

$\Leftarrow$ Assume $\lim_{n\to\infty} d(x_n, z) = 0$

To show: $\lim_{n\to\infty} x_n = z$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $x_n \in B_\varepsilon(z)$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $x_n \in B_\varepsilon(z)$.

Since $\lim_{n\to\infty} d(x_n, z) = 0$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $d(x_n, z) < \varepsilon$.

Let $\ell = N$.

To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $x_n \in B_\varepsilon(z)$

Assume $n \in \mathbb{Z}_{>0}$.

Since $d(x_n, z) < \varepsilon$ then $x_n \in B_\varepsilon(z)$. //