Let C be the Cantor set and $Q = \{x \in \mathbb{Q} \mid 0 \le x \le 1\}$.

(8)(a) To show: (aa) C is closed in X

(ab) C is not open in X

(ac) Q is not closed in X.

(ad) Q is not open in X.

(aa) Since $C = \left((\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \cup \cdots \right)^c$ then

C is the complement of a union of open intervals.

So C is the complement of an open set.

So C is closed.

(ab) $0 \in C$.

To show: 0 is not an interior point of C.

If $[0, a)$ is an open ~~set~~ interval containing 0

then $a > \frac{1}{3^N}$ for some $N \in \mathbb{Z}_{>0}$.

Since $(\frac{1}{3^N}, \frac{2}{3^N}) \subseteq C^c$ then $[0, a) \not\subseteq C$.

So 0 is not an interior point of C.

So C is not open.

(ac) To show: Q is not closed in X.

To show: Q^c is not open in X.

Let $a = \frac{1}{\sqrt{2}}$ so that $a \in Q^c$.

If $(a-\varepsilon, a+\varepsilon)$ is an open interval containing a

then $(a-\varepsilon, a+\varepsilon) \not\subseteq Q^c$ since $(a-\varepsilon, a+\varepsilon)$ contains a rational number.

So a is not an interior point of Q^c.

So Q^c is not open.

So Q is not closed.

(ad) To show: Q is not open.

~~Let~~ $0 \in Q$.

To show: 0 is not an interior point of Q.

Let $[0,a)$ be an open interval containing 0.

Then ~~then~~ $[0,a) \not\subseteq Q$ since $[0,a)$ contains an irrational number.

So 0 is not an interior point of Q.

So Q is not open.

(b) To show: (ba) C is nowhere dense in X

(bb) Q is dense in X.

(ba) To show: $(\overline{C})^\circ = \phi$.

Since C is closed $\overline{C} = C$.

To show: $C^\circ = \phi$.

To show: If $a \in C$ then a is not an interior point of C.

Assume $a \in C$.

To show: a is not an interior point of C.

If $(a-\varepsilon, a+\varepsilon)$ is an open interval containing then $(a-\varepsilon, a+\varepsilon)$ contains $\left(\frac{2k+1}{3^N}, \frac{2k+2}{3^N}\right)$ where $\frac{1}{3^N} < \varepsilon$ and $\frac{2k+1}{3^N} < a < \frac{2k+2}{3^N}$.

But $\left(\frac{2k+1}{3^N}, \frac{2k+2}{3^N}\right) \not\subseteq C$.

So a is not an interior point of C.

So $C^\circ = \emptyset$.

So C is nowhere dense in X.

(bb) To show: $\mathbb{Q}$ is dense in X.

To show: $\overline{\mathbb{Q}} = X$.

To show: If $a \in [0,1]$ then a is a close point to $\mathbb{Q}$.

Assume $a \in [0,1]$. Let $\varepsilon \in \mathbb{R}_{>0}$

Then every $(a-\varepsilon, a+\varepsilon)$ contains a rational number.

So $(a-\varepsilon, a+\varepsilon) \cap \mathbb{Q} \neq \emptyset$.

So a is a close point to $\mathbb{Q}$

So $\overline{\mathbb{Q}} = X$.

(8c) To show: (ca) C^c is dense in X

(cb) $\mathbb{Q}^c$ is dense in X.

(ca) To show: $\overline{C^c} = X$.

To show: If $x \in X$ then x is a close point to C^c

Assume $x \in X$.

To show: x is a close point to C^c.

Let $\varepsilon \in \mathbb{R}_{>0}$

To show: $(x-\varepsilon, x+\varepsilon) \cap C^c \neq \phi$.

Let $N \in \mathbb{Z}_{>0}$ such that $\frac{1}{3N} < \frac{\varepsilon}{3}$ and let $2k+1 \in \mathbb{Z}_{>0}$ be an odd integer such that $\frac{2k+1}{3N} < x < \frac{2k+2}{3N}$.

Then $\left(\frac{2k+1}{3N}, \frac{2k+2}{3N}\right) \subseteq (x-\varepsilon, x+\varepsilon)$ and $\left(\frac{2k+1}{3N}, \frac{2k+2}{3N}\right) \subseteq C^c$.

So $(x-\varepsilon, x+\varepsilon) \cap C^c \neq \phi$.

So x is a close point to C^c.

So $\overline{C^c} = X$ and C^c is dense in X.

(cb) To show: $\mathbb{Q}^c$ is dense in X.

To show: $\overline{\mathbb{Q}^c} = X$.

To show: If $x \in X$ then x is a close point to $\mathbb{Q}^c$.

Assume $x \in X$.

To show: x is a close point to $\mathbb{Q}^c$.

Let $\varepsilon \in \mathbb{R}_{>0}$.

To show: $(x-\varepsilon, x+\varepsilon) \cap \mathbb{Q}^c \neq \phi$.

Since $(x-\varepsilon, x+\varepsilon)$ contains an irrational number, $(x-\varepsilon, x+\varepsilon) \cap \mathbb{Q}^c \neq \phi$.

So x is a close point to $\mathbb{Q}^c$.

So $\overline{\mathbb{Q}^c} = X$ and $\mathbb{Q}^c$ is dense in X.

(8d) For this part we will use the following Theorem.

Theorem [Rubinstein, Theorem 6.5] and [Ram, Theorem 7.0.2].
If $A \subseteq \mathbb{R}$, with the standard topology, then
(*) A is compact if and only if A is closed and bounded.

To show: (da) C is compact.
(db) $\mathbb{Q}$ is not compact.

(db) By part (8a), $\mathbb{Q}$ is not closed in $\mathbb{R}$
So, by (*), $\mathbb{Q}$ is not compact.

(da) By part (8a), C is closed in $\mathbb{R}$.
Since $C \subseteq [0,1]$ then C is bounded (if $x, y \in C$ then $d(x,y) \le 1 < 2$).
So C is closed and bounded.
So, By (*), C is compact.

(8e) To show: (a) $\mathbb{Q}$ is totally disconnected
(b) C is totally disconnected.

(a) Let $x, y \in \mathbb{Q}$ with $x \neq y$. Assume $x < y$.

To show: There does not exist a connected set containing x and y.

Let E be a set containing x and y.

Let $z \in \mathbb{Q}^c$ with $x < z$ and $z < y$.

Let
$$A = (-\infty, z) \cap X \quad \text{and} \quad B = (z, \infty) \cap X.$$

Then $x \in A$ and $y \in B$ and $A \cap B = \phi$ ~~and $A \cap B = \phi$~~.

~~So~~ and $E \subseteq A \cup B$.

So E is not connected.

So there does not exist a connected set containing x and y.

So each connected component of $\mathbb{Q}$ contains only a single element.

So $\mathbb{Q}$ is totally disconnected.

(8e) part (b): To show: C is totally disconnected.

Let $x, y \in C$ with $x \neq y$. Assume $x < y$.

To show: There does not exist a connected subset of E containing x and y.

Let $E \subseteq C$ be a set containing x and y.

Let $N \in \mathbb{Z}_{>0}$ with $\frac{1}{3N} < \frac{y-x}{3}$ and let $k \in \mathbb{Z}_{>0}$ be the smallest positive integer such that

$$x < \frac{2k+1}{3N}. \quad \text{Then} \quad \frac{2k+2}{3N} < y.$$

Let
$$A = \left(-\infty, \frac{2k+1}{3N}\right) \cap C \quad \text{and} \quad B = \left(\frac{2k+2}{3N}, \infty\right) \cap C.$$

Then $x \in A$ and $y \in B$ and $A \cap B = \emptyset$. Since

$\left(\frac{2k+1}{3N}, \frac{2k+2}{3N}\right) \subseteq C^c$ then $E \subseteq A \cup B$.

So E is not connected

So there does not exist a connected set containing x and y.

So each connected component of C contains only a single element.

So C is totally disconnected.

(8e) $C = \left(\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \cdots \right)^c$.

Then $\mu(C^c) = \mu\left(\left(\frac{1}{3}, \frac{2}{3}\right)\right) + \mu\left(\left(\frac{1}{9}, \frac{2}{9}\right)\right) + \mu\left(\left(\frac{7}{9}, \frac{8}{9}\right)\right) + \cdots$

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \cdots$$

$$= \frac{1}{3} + 2\frac{1}{3^2} + 4\frac{1}{27} + 8\frac{1}{3^4} + \cdots$$

$$= \frac{1}{3} + 2\frac{1}{3^2} + 2^2\frac{1}{3^3} + 2^3\frac{1}{3^4} + \cdots$$

$$= \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right)$$

$$= \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{3}\,\frac{1}{\frac{1}{3}} = 1.$$

Since $\mu(C) + \mu(C^c) = \mu([0,1]) = 1$ then

$\mu(C) = 0$.

Using that $\mathbb{Q}$ is countable, $\mathrm{Card}(\mathbb{Q}) = \mathrm{Card}(\mathbb{Z}_{>0})$.
Enumerate the elements of $\mathbb{Q} = \{p_1, p_2, p_3, \ldots\}$.

Then $\mu(\mathbb{Q}) = \mu([p_1, p_1]) + \mu([p_2, p_2]) + \cdots$

$$= 0 + 0 + 0 + \cdots = 0.$$

Since

$\mu(\mathbb{Q}) + \mu(\mathbb{Q}^c) = \mu([0,1]) = 1$ then

$\mu(\mathbb{Q}^c) = 1$.

(8f) We will use the theorem that gives that if $f: S \to T$ is an injective function then $\mathrm{Card}(S) \le \mathrm{Card}(T)$. (see the Appendix of the notes.)

(fa) Since $\mathbb{Q} \subseteq \mathbb{Q}$ and $\mathrm{Card}(\mathbb{Q}) = \mathrm{Card}(\mathbb{Z})$ then

$$\mathrm{Card}(\mathbb{Q}) \le \mathrm{Card}(\mathbb{Z}) < \mathrm{Card}(\mathbb{R}).$$

So $\mathrm{Card}(\mathbb{Q}) \ne \mathrm{Card}(\mathbb{R})$.

(fb) Since $(\frac{1}{3}, \frac{2}{3}) \subseteq C^c$ and $\mathrm{Card}((\frac{1}{3}, \frac{2}{3})) = \mathrm{Card}(\mathbb{R})$

then $\mathrm{Card}(\mathbb{R}) = \mathrm{Card}((\frac{1}{3}, \frac{2}{3})) \le \mathrm{Card}(C^c) \le \mathrm{Card}(\mathbb{R})$.

So $\mathrm{Card}(C^c) = \mathrm{Card}(\mathbb{R})$.

(fc) Since $\mathrm{Card}(\mathbb{Q}) = \mathrm{Card}(\mathbb{Z})$, then $\mathbb{Q}$ is countable. If $\mathbb{Q}^c$ were countable then $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ would be countable. Since $\mathbb{R}$ is not countable, $\mathbb{Q}^c$ is not countable. So $\mathrm{Card}(\mathbb{Q}^c) = \mathrm{Card}(\mathbb{R})$.

(fd) For $\mathrm{Card}(\mathbb{R}) = \mathrm{Card}(C)$ see

Bourbaki, General Topology, Chapter IV §8 Ex. 9

Rudin, Principles of Mathematical Analysis, §2.44 and §2.43