(7a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

A function $f: X \to Y$ is <u>continuous</u> if f satisfies:

if $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.

(7b) Let $(X, \mathcal{X}_X)$ and $(Y, \mathcal{X}_Y)$ be uniform spaces.

A function $f: X \to Y$ is <u>uniformly continuous</u> if f satisfies:

if $D \in \mathcal{X}_Y$ then ~~there ex~~ $(f \times f)^{-1}(D) \in \mathcal{X}_X$.

(7c) Let $(X, \mathcal{X}_X)$ and $(Y, \mathcal{X}_Y)$ be uniform spaces

Let $f: X \to Y$ be a uniformly continuous function.

To show: f is continuous.

To show: If $a \in X$ and $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.

Assume $a \in X$ and $N \in \mathcal{N}(f(a))$.

By the definition of the uniform space topology

$$\mathcal{N}(f(a)) = \{ B_V(f(a)) \mid V \in \mathcal{X}_Y \} \text{ and}$$

$$\mathcal{N}(a) = \{ B_U(a) \mid U \in \mathcal{X}_X \},$$

where $B_V(f(a)) = \{ y \in Y \mid (y, f(a)) \in V \}$ and

$B_U(a) = \{ x \in X \mid (x, a) \in U \}$.

So there exists $V \in \mathcal{X}_Y$ such that

$$N = \{y \in Y \mid (y, f(a)) \in V\}.$$

To show: $f^{-1}(N) \in \mathcal{N}(a)$.

To show: There exists $U \in \mathcal{X}_X$ such that

$$f^{-1}(N) \supseteq B_U(a) = \{x \in X \mid (x,a) \in U\}.$$

Let $U = (f \times f)^{-1}(V)$.

Since f is uniformly continuous $U \in \mathcal{X}_X$.

To show: $f^{-1}(N) = B_U(a) = \{x \in X \mid (x,a) \in U\}$.

To show: (ca) $f^{-1}(N) \supseteq B_U(a)$

~~(cb) $f^{-1}(N) \subseteq B_U(a)$~~.

(ca) To show: $f(B_U(a)) \subseteq N$.

To show: If $x \in B_U(a)$ then $f(x) \in N$.

Assume $x \in B_U(a)$. Then $(x,a) \in U = (f \times f)^{-1}(V)$

So $(f \times f)(x,a) \in V$.

So $(f(x), f(a)) \in V$ and $f(x) \in B_V(f(a)) = N$.

So $f(B_U(a)) \subseteq N$.

So $f^{-1}(N) \supseteq B_U(a)$.

Since $B_U(a) \in \mathcal{N}(a)$ and $f^{-1}(N) \supseteq B_U(a)$

then $f^{-1}(N) \in \mathcal{N}(a)$.

So f is continuous. //

(d) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$ be a function. Show that f is continuous if and only if f satisfies:

(*) if $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

To show: (da) If f satisfies (*) then f is continuous.
(db) If f is continuous then f satisfies (*).

(da) Assume f satisfies (*)

To show: f is continuous.

To show: If V is open in Y then $f^{-1}(V)$ is open in X.

Assume V is open in Y.

To show: $f^{-1}(V)$ is open in X.

To show: If $x \in f^{-1}(V)$ then x is an interior point of $f^{-1}(V)$.

Assume $x \in f^{-1}(V)$

To show: x is an interior point of $f^{-1}(V)$.

Since $x \in f^{-1}(V)$ then $f(x) \in V$.

Since V is open there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(f(x)) \subseteq V$.

Since f satisfies $(*)$,

there exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

So there exists $\delta \in \mathbb{R}_{>0}$ such that

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x)).$$

So $f(B_\delta(x)) \subseteq V$.

So $B_\delta(x) \subseteq f^{-1}(V)$.

So x is an interior point of $f^{-1}(V)$.

So $f^{-1}(V)$ is open in X.

(db) To show: If f is continuous then f satisfies $(*)$

Assume f is continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $y \in X$ and $d_X(x,y) < \delta$ then $d_X(f(x), f(y)) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$.

Since f is continuous and $B_\varepsilon(f(x))$ is open in Y

$f^{-1}(B_\varepsilon(f(x)))$ is open in X.

We know $x \in f^{-1}(B_\varepsilon(f(x)))$ and, since $f^{-1}(B_\varepsilon(f(x)))$ is open, x is an interior point of $f^{-1}(B_\varepsilon(f(x)))$.

So there exists $B_\delta(x)$ with $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

So there exists $\delta \in \mathbb{R}_{>0}$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

So there exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

(e) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$ be a function. Show that f is uniformly continuous if and only if f satisfies

(**) if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

To show: (ea) If f satisfies (**) then f is uniformly continuous.

(eb) If f is uniformly continuous then f satisfies (**).

(ea) Assume f satisfies (**).

To show: f is uniformly continuous.

To show: If $V \in \mathcal{X}_Y$ then $(f \times f)^{-1}(V) \in \mathcal{X}_X$.

Assume $V \in \mathcal{X}_Y$.

Then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon \subseteq V$.

To show: $(f \times f)^{-1}(V) \in \mathcal{X}_X$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that $B_\delta \subseteq (f \times f)^{-1}(V)$

Using that f satisfies (**) we know

there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x, y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

So there exists $\delta \in \mathbb{R}_{>0}$ such that $(f\times f)(B_\delta) \subseteq B_\varepsilon$.

So there exists $\delta \in \mathbb{R}_{>0}$ such that $B_\delta \subseteq (f\times f)^{-1}(B_\varepsilon)$.

So there exists $\delta \in \mathbb{R}_{>0}$ such that $B_\delta \subseteq (f\times f)^{-1}(V)$.

So $(f\times f)^{-1}(V) \in \mathcal{X}_X$.

So f is uniformly continuous.

(ab) To show: If f is uniformly continuous then f satisfies $(**)$.

Assume f is uniformly continuous.

To show: ~~If~~ $\varepsilon \in \mathbb{R}_{>0}$ ~~and $x \in$~~ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that $\cancel{B_\delta}$ if $(x,y) \in B_\delta$ then $(f(x), f(y)) \in B_\varepsilon$.

Since f is uniformly continuous ~~we know~~ and $B_\varepsilon \in \mathcal{X}_Y$ we know ~~there exists $B_\delta \subseteq$~~ $(f\times f)^{-1}(B_\varepsilon) \in \mathcal{X}_X$.

So there exists $\delta \in \mathbb{R}_{>0}$ such that $B_\delta \subseteq (f\times f)^{-1}(B_\varepsilon)$

So there exists $\delta \in \mathbb{R}_{>0}$ such that $(f\times f)(B_\delta) \subseteq B_\varepsilon$

So there exists $\delta \in \mathbb{R}_{>0}$ such that if $(x,y) \in B_\delta$ then $(f(x), f(y)) \in B_\varepsilon$.

So f satisfies $(**)$.