(6a) To show: (aa) If $(f_1, f_2, \ldots)$ converges pointwise to f then $(f_1, f_2, \ldots)$ satisfies

if $x \in X$ then $\lim_{n\to\infty} d(f_n(x), f(x)) = 0$.

(ab) If $(f_1, f_2, \ldots)$ satisfies

if $x \in X$ then $\lim_{n\to\infty} d(f_n(x), f(x)) = 0$

then $(f_1, f_2, \ldots)$ converges pointwise to f.

(aa) Assume $(f_1, f_2, \ldots)$ converges pointwise to f.

To show: If $x \in X$ then $\lim_{n\to\infty} d(f_n(x), f(x)) = 0$

Assume $x \in X$.

To show: $\lim_{n\to\infty} d(f_n(x), f(x)) = 0$

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $d(f_n(x), f(x)) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $d(f_n(x), f(x)) < \varepsilon$.

This is true by the definition of "$(f_1, f_2, \ldots)$ converges pointwise to f".

(ab) Assume $(f_1, f_2, \ldots)$ satisfies

if $x \in X$ then $\lim_{n\to\infty} d(f_n(x), f(x)) = 0$.

To show: $(f_1, f_2, \ldots)$ converges pointwise to f.

To show: If $x\in X$ and $\varepsilon\in\mathbb{R}_{>0}$ then ~~there~~ exists $N\in\mathbb{Z}_{>0}$ such that if $n\in\mathbb{Z}_{\ge N}$ then $d(f_n(x), f(x))<\varepsilon$.

Assume $x\in X$ and $\varepsilon\in\mathbb{R}_{>0}$.

To show: There exists $N\in\mathbb{Z}_{>0}$ such that if $n\in\mathbb{Z}_{\ge N}$ then $d(f_n(x), f(x))<\varepsilon$.

We know: $\lim\limits_{n\to\infty} d(f_n(x), f(x)) = 0$.

So, by the definition of the limit, there exists $N\in\mathbb{Z}_{>0}$ such that if $n\in\mathbb{Z}_{\ge N}$ then $d(f_n(x), f(x))<\varepsilon$..

(6b) To show: (6a) If $(f_1, f_2, \ldots)$ converges uniformly to f then $\lim\limits_{n\to\infty} d_\infty(f_n, f)=0$

(6b) If $(f_1, f_2, \ldots)$ satisfies $\lim\limits_{n\to\infty} d_\infty(f_n, f)=0$ then $(f_1, f_2, \ldots)$ converges uniformly to f.

(6a) Assume $(f_1, f_2, \ldots)$ converges uniformly to f.

To show: $\lim\limits_{n\to\infty} d_\infty(f_n, f)=0$.

To show: If $\varepsilon\in\mathbb{R}_{>0}$ then there exists $N\in\mathbb{Z}_{>0}$ such that if $n\in\mathbb{Z}_{\ge N}$ then $d_\infty(f_n, f)<\varepsilon$.

Assume $\varepsilon\in\mathbb{R}_{>0}$

To show: There exists $N\in\mathbb{Z}_{>0}$ such that if $n\in\mathbb{Z}_{\ge N}$ then $\sup\{\rho(f_n(x), f(x)) \mid x\in X\}<\varepsilon$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geq N}$ and $x \in X$ then $\rho(f_n(x), f(x)) < \varepsilon$.

This is true by the definition of "$(f_1, f_2, \ldots)$ converges uniformly to f".

(bb) Assume $(f_1, f_2, \ldots)$ satisfies $\lim_{n \to \infty} d_\infty(f_n, f) = 0$.

To show: $(f_1, f_2, \ldots)$ converges uniformly to f.

We know: $\lim_{n \to \infty} d_\infty(f_n, f) = 0$.

So, if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geq N}$ then $d_\infty(f_n, f) < \varepsilon$.

So, if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geq N}$ then $\sup\{\rho(f_n(x), f(x)) \mid x \in X\} < \varepsilon$.

So, if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geq N}$ and $x \in X$ then $\rho(f_n(x), f(x)) < \varepsilon$.