(5) Let $(V, \| \|)$ and $(W, \| \|)$ be normed vector spaces.

The space of bounded operators from V to W is

$$B(V,W) = \left\{ \begin{matrix} \text{linear transformations} \\ T: V \to W \end{matrix} \;\middle|\; \|T\| < \infty \right\}$$

where

$$\|T\| = \sup \left\{ \frac{\|Tv\|}{\|v\|} \;\middle|\; v \in V \right\}$$

To show: If W is complete then $B(V,W)$ is complete.

Assume W is complete

To show: $B(V,W)$ is complete.

To show: If $T_1, T_2, \ldots$ is a Cauchy sequence in $B(V,W)$
then $T_1, T_2, \ldots$ converges.

Assume $T_1: V \to W$, $T_2: V \to W, \ldots$ is a Cauchy sequence
in $B(V,W)$

To show: There exists $T: V \to W$ with $T \in B(V,W)$
such that $\lim_{n \to \infty} T_n = T$.

Define $T: V \to W$ by

$$T(x) = \lim_{n \to \infty} T_n(x).$$

To show: (a) If $x \in V$ then $T(x)$ exists

(b) $T \in B(V, W)$

(c) $\lim_{n\to\infty} T_n = T$.

(a) Assume $x \in V$.

To show: $T_1(x), T_2(x), \ldots$ converges in W.

Since W is complete,

To show: $T_1(x), T_2(x), \ldots$ is Cauchy.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $\|T_r(x) - T_s(x)\| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Using that $T_1, T_2, \ldots$ is Cauchy, let N be such that if $r, s \in \mathbb{Z}_{\geq N}$ then $\|T_r - T_s\| < \varepsilon/\|x\|$.

To show: If $r, s \in \mathbb{Z}_{\geq N}$ then $\|T_r(x) - T_s(x)\| < \varepsilon$

Assume $r, s \in \mathbb{Z}_{\geq N}$

To show: $\|T_r(x) - T_s(x)\| < \varepsilon$

$$\|T_r(x) - T_s(x)\| \leq \|T_r - T_s\| \cdot \|x\| < \frac{\varepsilon}{\|x\|} \cdot \|x\| = \varepsilon.$$

So $T_1(x), T_2(x), \ldots$ is Cauchy and, since W is complete, $T_1(x), T_2(x), \ldots$ converges.

So $T(x) = \lim_{n\to\infty} T_n(x)$ exists.

(b) To show: $T \in B(V, W)$.

To show: (ba) T is a linear transformation.

(bb) $\|T\| < \infty$

(ba) To show: (baa) If $x_1, x_2 \in V$ then $T(x_1+x_2) = T(x_1) + T(x_2)$

(bab) If $c \in \mathbb{K}$ and $x \in V$ then $T(cx) = cT(x)$.

(baa) Assume $x_1, x_2 \in V$.

To show: $T(x_1+x_2) = T(x_1) + T(x_2)$

Since each T_n is a linear transformation and since $\begin{matrix} + : W \times W \to W \\ (w_1, w_2) \mapsto w_1 + w_2 \end{matrix}$ is continuous,

$$T(x_1+x_2) = \lim_{n\to\infty} T_n(x_1+x_2) = \lim_{n\to\infty} (T_n(x_1) + T_n(x_2))$$

$$= \lim_{n\to\infty} T_n(x_1) + \lim_{n\to\infty} T_n(x_2)$$

$$= T(x_1) + T(x_2)$$

(bab) Assume $c \in \mathbb{K}$ and $x \in V$.

To show: $T(cx) = cT(x)$.

Since each T_n is a linear transformation and scalar multiplication $\begin{matrix} \mathbb{K} \times W \to W \\ (c, w) \mapsto cw \end{matrix}$ is continuous,

$$T(cx) = \lim_{n\to\infty} T_n(cx) = \lim_{n\to\infty} cT_n(x) = c \lim_{n\to\infty} T_n(x) = cT(x).$$

So T is a linear transformation.

(bb) To show: $\|T\| < \infty$

To show: $\|T\| = \sup\left\{ \frac{\|Tx\|}{\|x\|} \mid x \in V \right\}$ exists in $\mathbb{R}_{\geq 0}$.

Since $\| \, \| : W \to \mathbb{R}_{\geq 0}$ is continuous,

$$\|Tx\| = \|\lim_{n\to\infty} T_n(x)\| = \lim_{n\to\infty} \|T_n(x)\|$$

$$\leq \lim_{n\to\infty} \|T_n\| \cdot \|x\| = \|x\| \cdot \left(\lim_{n\to\infty} \|T_n\|\right)$$

Since $T_1, T_2, \ldots$ is a Cauchy sequence and $\|T_r\| - \|T_s\| \leq \|T_r - T_s\|$, the sequence $\|T_1\|, \|T_2\|, \ldots$ is Cauchy.

Since $\mathbb{R}_{\geq 0}$ is complete, $\lim_{n\to\infty} \|T_n\|$ exists.

So

$$\|T\| = \sup\left\{ \frac{\|Tx\|}{\|x\|} \mid x \in V \right\} \leq \lim_{n\to\infty} \|T_n\|$$

and the right hand side exists in $\mathbb{R}_{\geq 0}$.

So $\|T\| < \infty$

So $T \in B(V, W)$.

(c) To show: $\lim_{n\to\infty} T_n = T$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\|T - T_n\| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

Using that the sequence $T_1, T_2, \ldots$ is Cauchy let $N \in \mathbb{Z}_{>0}$ be such that if $m, n \in \mathbb{Z}_{\geq N}$ then $\|T_m - T_n\| < \frac{\varepsilon}{2}$.

To show: If $n \in \mathbb{Z}_{\geq N}$ then $\|T - T_n\| < \varepsilon$.

Assume $n \in \mathbb{Z}_{\geq N}$.

To show: $\|T - T_n\| < \varepsilon$.

Since $\|\ \|: W \to \mathbb{R}_{\geq 0}$ is continuous,

$$\|T(x) - T_n(x)\| = \|\lim_{m\to\infty} (T_m(x) - T_n(x))\|$$

$$= \lim_{m\to\infty} \|T_m(x) - T_n(x)\|.$$

Since $\|T_m - T_n\| < \frac{\varepsilon}{2}$ for $m \in \mathbb{Z}_{\geq N}$ then

$$\|T_m(x) - T_n(x)\| < \frac{\varepsilon}{2}\|x\| \text{ for } m \in \mathbb{Z}_{\geq N} \text{ and}$$

thus

$$\|T(x) - T_n(x)\| = \lim_{m\to\infty} \|T_m(x) - T_n(x)\| \leq \lim_{m\to\infty} \frac{\varepsilon}{2}\|x\| = \frac{\varepsilon}{2}\|x\|.$$

So $\|T - T_n\| \leq \frac{\varepsilon}{2} < \varepsilon$

So $\lim_{n\to\infty} \|T - T_n\| = 0$.

So $\lim_{n\to\infty} T_n = T$. //