(4a) To show: If $p \le q$ then $\ell^p \subseteq \ell^q$.

Assume $p \le q$.

To show: $\ell^p \subseteq \ell^q$

To show: If $x = (x_1, x_2, \ldots) \in \ell^p$ then $x \in \ell^q$.

Assume $x = (x_1, x_2, \ldots) \in \ell^p$.

Then $\displaystyle\sum_{i=1}^{\infty} |x_i|^p < \infty$.

There exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\ge N}$ then $|x_n| < 1$.

Since $p \le q$ and $|x_n| < 1$ then

$$|x_n|^p \ge |x_n|^q$$

To show: $x = (x_1, x_2, \ldots) \in \ell^q$.

To show: $\displaystyle\sum_{i=1}^{\infty} |x_i|^q < \infty$.

$$\sum_{i=1}^{\infty} |x_i|^q = \sum_{i=1}^{N} |x_i|^q + \sum_{i=N+1}^{\infty} |x_i|^q$$

$$\le \left(\sum_{i=1}^{N} |x_i|^q\right) + \sum_{i=N+1}^{\infty} |x_i|^p < \infty.$$

So $x = (x_1, x_2, \ldots) \in \ell^q$.

So $\ell^p \subseteq \ell^q$.

(4b) To show: If $p \neq q$ then $\ell^p \neq \ell^q$.

Assume $p \neq q$ and $p < q$.

We know $\ell^p \subseteq \ell^q$.

To show: There exists $y = (y_1, y_2, \ldots) \in \ell^q$ such that $y \notin \ell^p$.

Let $y = \left(\left(\frac{1}{1}\right)^{1/p}, \left(\frac{1}{2}\right)^{1/p}, \left(\frac{1}{3}\right)^{1/p}, \ldots \right)$

To show: $y \in \ell^q$ and $y \notin \ell^p$.

To show: (ba) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{q/p} < \infty$

(bb) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{p/p}$ diverges

(bb) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{p/p} = \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots$

$\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$

So $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{p/p}$ diverges.

So $y \notin \ell^p$.

(ba) To show: $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{q/p}$ converges.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{q/p} = 1 + \left(\frac{1}{2^{q/p}} + \frac{1}{3^{q/p}}\right) + \left(\frac{1}{4^{q/p}} + \frac{1}{5^{q/p}} + \frac{1}{6^{q/p}} + \frac{1}{7^{q/p}}\right) + \cdots$$

$$< 1 + \frac{2}{2^{q/p}} + \frac{4}{4^{q/p}} + \cdots$$

$$= 1 + \frac{1}{2^{q/p-1}} + \frac{1}{4^{q/p-1}} + \frac{1}{8^{q/p-1}} + \cdots$$

$$= 1 + \frac{1}{2^{q/p-1}} + \left(\frac{1}{2^{q/p-1}}\right)^2 + \left(\frac{1}{2^{q/p-1}}\right)^3 + \cdots$$

$$= \frac{1}{1 - \frac{1}{2^{q/p-1}}} = \frac{2^{q/p-1}}{2^{q/p-1} - 1}.$$

So $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{q/p}$ converges.

So $y = \left(\frac{1}{1^{1/p}}, \frac{1}{2^{1/p}}, \frac{1}{3^{1/p}}, \ldots\right) \in l^q$.

So $l^q \neq l^p$. //