(3a) To show: (aa) If $E \subseteq X$ then $\overline{E^c} = (E^\circ)^c$

(ab) If $E \subseteq X$ then $(E^c)^\circ = (\overline{E})^c$

(aa) Assume $E \subseteq X$.

To show: $\overline{E^c} = (E^\circ)^c$.

To show: (aaa) $(E^\circ)^c \supseteq E^c$ and $(E^\circ)^c$ is closed.

(aab) If F is closed and $F \supseteq E^c$ then $F \supseteq (E^\circ)^c$

(aaa) $(E^\circ)^c$ is closed since E° is open

$(E^\circ)^c \supseteq E^c$ since $E^\circ \subseteq E$.

(aab) Assume F is closed and $F \supseteq E^c$.

To show: $F \supseteq (E^\circ)^c$.

To show: $F^c \subseteq E^\circ$.

Since $F \supseteq E^c$ then $F^c \subseteq E$.

Since F is closed, F^c is open and so $F^c \subseteq E^\circ$.

So $\overline{E^c} = (E^\circ)^c$

(ab) To show: If $E \subseteq X$ then $(E^c)^\circ = (\overline{E})^c$

Assume $E \subseteq X$ and let $F = E^c$

To show: $F^\circ = (\overline{F^c})^c$.

To show: $(F^\circ)^c = \overline{F^c}$.

This is true, by (aa).

So $(E^c)^\circ = (\overline{E})^c$.

(3b) To show: (a) If E is an open dense subset of X then E^c is nowhere dense in X.

(b) If $E \subseteq X$ and E^c is nowhere dense in X then E is open dense in X.

(a) Assume $E \subseteq X$, E is open in X, $\overline{E} = X$.

To show: $(\overline{E^c})^\circ = \phi$ (definition of nowhere dense).

We know

$$\phi = X^c = (\overline{E})^c = (E^c)^\circ, \text{ by } (4a).$$

Since E is open, E^c is closed and $E^c = \overline{E^c}$.

So $\phi = (E^c)^\circ = (\overline{E^c})^\circ$.

So E^c is nowhere dense in X.

(b) Assume $E \subseteq X$ and $(\overline{E^c})^\circ = \phi$.

To show: E is open and $\overline{E} = X$.

Since $X = ((\overline{E^c})^\circ)^c = \overline{(\overline{E^c})^c} = \overline{((E^\circ)^c)^c} = \overline{E^\circ}$,

it follows that $\overline{E} = X$ and E is dense in X.

(3c) Let $U_1, U_2, \ldots$ be open dense subsets of a topological space. 3

To show: (ca) If $\bigcup_{i \in \mathbb{Z}_{>0}} U_i$ is dense in X then

$$\bigcap_{i \in \mathbb{Z}_{>0}} (U_i)^c \text{ has empty interior.}$$

(cb) If $\bigcap_{i \in \mathbb{Z}_{>0}} (U_i)^c$ has empty interior then

$$\bigcup_{i \in \mathbb{Z}_{>0}} U_i \text{ is dense in } X.$$

(ca) Assume $\overline{\left(\bigcup_i U_i\right)} = X$.

To show: $\left(\bigcap_i U_i^c\right)^\circ = \phi$.

To show: $\left(\left(\bigcap_i U_i^c\right)^\circ\right)^c = X$.

To show: $\overline{\left(\bigcap_i U_i^c\right)^c} = X$.

We know

$$\left(\bigcap_i U_i^c\right)^c = \bigcup_i (U_i^c)^c = \bigcup_i U_i$$

So $\overline{\left(\bigcup_i U_i\right)} = X$ then $\overline{\left(\bigcap_i U_i^c\right)^c} = X$.

(cb) Assume $\left(\bigcap_i U_i^c\right)^\circ = \phi$

To show: $\overline{\left(\bigcup_i U_i\right)} = X$.

$$\overline{\left(\bigcup_i U_i\right)} = \overline{\left(\bigcap_i U_i^c\right)^c} = \left(\left(\bigcap_i U_i^c\right)^\circ\right)^c = \phi^c = X.$$

(3d) Show that an open set in $X \times Y$ cannot be expected to be of the form $A \times B$ with A open in X and B open in Y.

To show: There exists an open set Z in $X \times Y$ such that $Z \neq A \times B$ with A open in X and B open in Y.

Let $X = \{0,1\}$ with $\mathcal{T}_X = \{\phi, \{0\}, \{1\}, \{0,1\}\}$

$Y = \{0,1\}$ with $\mathcal{T}_Y = \{\phi, \{0\}, \{1\}, \{0,1\}\}$

Let $Z = \{(0,0), (1,1)\} \subseteq X \times Y$.

$\mathcal{T}_X \times \mathcal{T}_Y = \{A \times B \mid A \in \mathcal{T}_X, B \in \mathcal{T}_Y\}$

$$= \left\{ \begin{array}{l} \phi, \{(0,0)\}, \{(0,1)\}, \{(0,0),(0,1)\} \\ \{(1,0)\}, \{(1,1)\}, \{(1,0),(1,1)\} \\ \{(0,0),(1,0)\}, \{(0,1),(1,1)\} \\ \{(0,0),(0,1),(1,0),(1,1)\} \end{array} \right\}$$

is not a topology. The topology generated by $\mathcal{T}_X \times \mathcal{T}_Y$ is the discrete topology on $X \times Y$,

$\mathcal{T}_{X \times Y} = \{\text{subsets of } X \times Y\}$.

$Z = \{(0,0), (1,1)\}$ is a set in $\mathcal{T}_{X \times Y}$ which is not in $\mathcal{T}_X \times \mathcal{T}_Y$.

So $Z \neq A \times B$ for any $A \in \mathcal{T}_X$ and $B \in \mathcal{T}_Y$.

(3e) Let $A \subseteq X$ and $B \subseteq Y$.

To show: (a) $\overline{A} \times \overline{B} = \overline{A \times B}$

(b) $A^\circ \times B^\circ = (A \times B)^\circ$.

(b) To show: (ba) $A^\circ \times B^\circ \subseteq (A \times B)^\circ$

(bb) $(A \times B)^\circ \subseteq A^\circ \times B^\circ$.

(ba) Assume $(x,y) \in A^\circ \times B^\circ$.

To show: $(x,y) \in (A \times B)^\circ$.

Since $(x,y) \in A^\circ \times B^\circ$, x is an interior point of A and y is an interior point of B

So there exists a neighborhood N_x of x with $N_x \subseteq A$.
and a neighborhood N_y of y with $N_y \subseteq B$.

Next let U_x be open in X with $x \in U_x \subseteq N_x \subseteq A$.
and U_y open in Y with $y \in U_y \subseteq N_y \subseteq B$.

Then $U_x \times U_y$ is open in $X \times Y$ and $U_x \times U_y \subseteq A \times B$.

Since $(x,y) \in U_x \times U_y$ then (x,y) is an interior point of $A \times B$.

So $(x,y) \in (A \times B)^\circ$

(bb) Assume $(x,y) \in (A \times B)^\circ$

To show: $x \in A^\circ$ and $y \in B^\circ$.

Let N be a neighborhood of (x,y) with $N \subseteq A \times B$.

Then there exists N_x, a neighborhood of $x \in X$,
and N_y, a neighborhood of y in Y
such that $N_x \times N_y \subseteq N$.

Since $N_x \times N_y \subseteq N \subseteq A \times B$ then

$x \in N_x \subseteq A$ and $y \in N_y \subseteq B$.

So x is an interior point of A, and
y is an interior point of B.

(a) To show: $\bar{A} \times \bar{B} = \overline{A \times B}$

To show: (aa) $\bar{A} \times \bar{B} \subseteq \overline{A \times B}$
(ab) $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$

(aa) Assume $(x,y) \in \bar{A} \times \bar{B}$

To show: $(x,y) \in \overline{A \times B}$

To show: (x,y) is a close point of $A \times B$.

Let N be a neighborhood of (x,y) in $X \times Y$.

By the definition of the product topology on $X \times X$ there exist N_x, a neighborhood of $x \in X$, and
N_y, a neighborhood of $y \in Y$,
such that $N_x \times N_y \subseteq N$.

Since $x \in \bar{A}$ there exists $a \in A$ with $a \in N_x$.
Since $y \in \bar{B}$ there exists $b \in B$ with $b \in N_y$
So $(a,b) \in N_x \times N_y \subseteq N$ and $(a,b) \in A \times B$
So (x,y) is a close point of $A \times B$.
So $(x,y) \in \overline{A \times B}$
So $\bar{A} \times \bar{B} \subseteq \overline{A \times B}$.

(3e) part (ab): To show: $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$.

Assume $(x,y) \in \overline{A \times B}$

To show: $(x,y) \in \overline{A} \times \overline{B}$

To show: $x \in \overline{A}$ and $y \in \overline{B}$

Let N_x be a neighborhood of $x \in X$ and

let N_y be a neighborhood of y in Y.

Then $N_x \times N_y$ is a neighborhood of (x,y) in $X \times Y$.

Since (x,y) is a close point of $A \times B$,

there exists $(a,b) \in A \times B$ with $(a,b) \in N_x \times N_y$

So $a \in N_x$ and $b \in N_y$ and $a \in A$ and $b \in B$.

So x is a close point of A and

y is a close point of B.

So $x \in \overline{A}$ and $y \in \overline{B}$.

So $(x,y) \in \overline{A} \times \overline{B}$