(2a) Let $(V, \langle , \rangle)$ be a positive definite inner product space and let $\| \; \|: V \to \mathbb{R}_{\geq 0}$ be given by $\|v\|^2 = \langle v, v \rangle$.

To show: If $x, y \in V$ then $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.

Assume $x, y \in V$. Let $W = \mathbb{K}\text{-span}\{x, y\}$.

To show: $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.

Case 1: $\dim(W) = 0$.

Then $x = 0$ and $y = 0$ and $|\langle x, y\rangle| = 0 = \|x\| \cdot \|y\|$.

Case 2: $\dim(W) = 1$.

Then there exists $c \in \mathbb{K}$ such that $y = cx$ and

$|\langle x, y\rangle| = |\langle x, cx\rangle| = |c| \cdot \|x\|^2 = \|x\| \cdot \|cx\| = \|x\| \cdot \|y\|$.

Case 3: $\dim(W) = 2$.

Then (x, y) is a basis of W and, with respect to this basis, the matrix of $\langle , \rangle$ on W is

$$\begin{pmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{pmatrix} = \begin{pmatrix} \|x\|^2 & \langle x, y\rangle \\ \langle y, x\rangle & \|y\|^2 \end{pmatrix}$$

Do Gram-Schmidt and change basis from $\{x, y\}$ to $\{w_1, w_2\}$ where

$$w_1 = \frac{x}{\|x\|} \quad \text{and} \quad w_2 = \frac{y - \langle w_1, y\rangle w_1}{\|y - \langle w_1, y\rangle w_1\|}.$$

then the matrix of $\langle\, , \rangle$ with respect to the basis $\{w_1, w_2\}$ is the identity.

If

$$x = a_1 w_1 + a_2 w_2 \quad \text{and} \quad y = b_1 w_1 + b_2 w_2$$

then

$$|\langle x,y\rangle| = |a_1 b_1 + a_2 b_2| \quad \text{and}$$

$$\|x\|^2 = a_1^2 + a_2^2 \quad \text{and} \quad \|y\|^2 = b_1^2 + b_2^2.$$

Since $0 \le (a_1 b_2 - a_2 b_1)^2 = -2a_1 b_1 a_2 b_2 + a_1^2 b_2^2 + a_2^2 b_1^2$,

$$\begin{aligned} |\langle x,y\rangle|^2 = (a_1 b_1 + a_2 b_2)^2 &= a_1^2 b_1^2 + 2a_1 b_1 a_2 b_2 + a_2^2 b_2^2 \\ &\le a_1^2 b_1^2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_2^2 b_2^2 \\ &= (a_1^2 + a_2^2)(b_1^2 + b_2^2) = \|x\|^2 \cdot \|y\|^2 \end{aligned}$$

Thus $|\langle x,y\rangle| \le \|x\| \cdot \|y\|$

(2b) To show: If $x, y \in V$ then $\|x+y\| \le \|x\| + \|y\|$.

Assume $x, y \in V$.

By (a), $\mathrm{Re}(\langle x,y\rangle) \le |\langle x,y\rangle| \le \|x\| \cdot \|y\|$ so that

$$\begin{aligned} \|x+y\|^2 = \langle x+y, x+y\rangle &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle) \\ &\le \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| = (\|x\| + \|y\|)^2. \ // \end{aligned}$$

(2c) Show that if $x, y \in V$ and $\langle x,y\rangle = 0$ then $\|x\|^2 + \|y\|^2 = \|x+y\|^2$.

Assume $x, y \in V$ and $\langle x,y\rangle = 0$.

To show: $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

$$
\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\
&= \|x\|^2 + 0 + \overline{\langle x,y\rangle} + \|y\|^2 \\
&= \|x\|^2 + 0 + 0 + \|y\|^2 = \|x\|^2 + \|y\|^2.
\end{aligned}
$$

(2d) Show that if $x, y \in V$ then
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Assume $x, y \in V$.

To show: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

$$
\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\
&\quad + \langle x,x\rangle + \langle x,-y\rangle + \langle -y,x\rangle + \langle -y,-y\rangle \\
&= \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2 \\
&\quad + \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2 \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}
$$

(2e) Let $(V, \|\ \|)$ be a normed vector space over $\mathbb{C}$ that satisfies

if $x, y \in V$ then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Define $\langle , \rangle : V \times V \to \mathbb{C}$ by

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

Show that $(V, \langle\ \rangle)$ is a positive definite Hermitian inner product space.

To show: (a) If $v_1, v_2 \in V$ then $\langle v_2, v_1 \rangle = \overline{\langle v_1, v_2 \rangle}$

(b) If $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2, v_3 \in V$ then

$\langle c_1 v_1 + c_2 v_2, v_3 \rangle = c_1 \langle v_1, v_3 \rangle + c_2 \langle v_2, v_3 \rangle$.

(c) If $v \in V$ and $\langle v, v \rangle = 0$ then $v = 0$.

(d) If $v \in V$ ~~and~~ then $\langle v, v \rangle \in \mathbb{R}_{\geq 0}$.

(d) To show: If $v \in V$ then $\langle v, v \rangle \in \mathbb{R}_{\geq 0}$

Assume $v \in V$.

To show: $\langle v, v \rangle \in \mathbb{R}_{\geq 0}$.

$$\begin{aligned}\langle v, v \rangle &= \frac{1}{4}\left(\|v+v\|^2 - \|v-v\|^2 + i\|v+iv\|^2 - i\|v-iv\|^2\right) \\ &= \frac{1}{4}\left(\|2v\|^2 - 0 + i\|(1+i)v\|^2 - i\|(1-i)v\|^2\right) \\ &= \frac{1}{4}\left(4 + i|1+i|^2 - i|1-i|^2\right)\|v\|^2\end{aligned}$$

$= \frac{1}{4}\left(4 + i(1+i)(1-i) - i(1-i)(1+i)\right) \|v\|^2 = \|v\|^2 \in \mathbb{R}_{\geq 0}.$

(c) To show: If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$.

Assume $v \in V$ and $\langle v, v\rangle = 0$.

To show: $v = 0$.

We know: $\|v\|^2 = \langle v, v\rangle = 0$ (from the computation in part (d))

So $\|v\| = 0$.

So $v = 0$.

(a) To show: If $v_1, v_2 \in V$ then $\langle v_2, v_1\rangle = \overline{\langle v_1, v_2\rangle}$

Assume $v_1, v_2 \in V$.

To show: $\langle v_2, v_1\rangle = \overline{\langle v_1, v_2\rangle}$

$$\langle v_2, v_1\rangle = \frac{1}{4}\left(\|v_2+v_1\|^2 - \|v_2 - v_1\|^2 + i\|v_2 + iv_1\|^2 - i\|v_2 - iv_1\|^2\right)$$

$$= \frac{1}{4}\Big(\|v_1+v_2\|^2 - \|(-1)(v_1-v_2)\|^2 + i\|(+i)(v_1 - iv_2)\|^2 - i\|(-i)(v_1+iv_2)\|^2\Big)$$

$$= \frac{1}{4}\left(\|v_1+v_2\|^2 - \|v_1-v_2\|^2 + i|i|^2\|v_1 - iv_2\|^2 - i|-i|^2\|v_1+iv_2\|^2\right)$$

$$= \frac{1}{4}\left(\|v_1+v_2\|^2 - \|v_1-v_2\|^2 - \left(i\|v_1+iv_2\|^2 - i\|v_1 - iv_2\|^2\right)\right)$$

$$= \overline{\langle v_1, v_2\rangle}.$$

(b) To show: If $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, v_3 \in V$ then

$\langle c_1 v_1 + c_2 v_2, v_3\rangle = c_1\langle v_1, v_3\rangle + c_2\langle v_2, v_3\rangle$

To show: (6a) If $v_1, v_2, v_3 \in V$ then

$$\langle v_1+v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle.$$

(6b) If $c \in \mathbb{C}$ and $v_1, v_2 \in V$ then

$$\langle cv_1, v_2\rangle = c\langle v_1, v_2\rangle.$$

(6a) First write

$$\langle x,y\rangle = \tfrac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\left(\|x+iy\|^2 - i\|x-iy\|^2\right)\right)$$

$$= \tfrac{1}{4}\left(\begin{array}{l}\|x+y\|^2 + \|x+y\|^2 - 2\|x\| - 2\|y\|^2 \\ \quad + i\left(\|x+iy\|^2 + \|x+iy\|^2 - 2\|x\|^2 - 2\|iy\|^2\right)\end{array}\right)$$

$$= \tfrac{1}{2}\left(\|x+y\|^2 - \|x\| - \|y\|^2 + i\left(\|x+iy\|^2 - \cancel{2}\|x\|^2 - \cancel{2}\|y\|^2\right)\right)$$

Writing $\mathrm{Re}(\langle x,y\rangle) = \tfrac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$

and $\mathrm{Im}(\langle x,y\rangle) = \tfrac{1}{2}\left(\|x+iy\|^2 - \|x\|^2 - \|iy\|^2\right)$

the computation ~~parts~~ (2f) part (6a) gives

$$\mathrm{Re}(\langle v_1+v_2, v_3\rangle) = \mathrm{Re}(\langle v_1, v_3\rangle) + \mathrm{Re}(\langle v_2, v_3\rangle) \quad \text{and}$$

$$\mathrm{Im}(\langle v_1+v_2, v_3\rangle) = \mathrm{Im}(\langle v_1, v_3\rangle) + \mathrm{Im}(\langle v_2, v_3\rangle).$$

Thus

$$\langle v_1+v_2, v_3\rangle = \mathrm{Re}(\langle v_1+v_2, v_3\rangle) + i\,\mathrm{Im}(\langle v_1+v_2, v_3\rangle)$$

$$= \mathrm{Re}(\langle v_1, v_3\rangle) + i\,\mathrm{Im}(\langle v_1, v_3\rangle) + \mathrm{Re}(\langle v_2, v_3\rangle) + i\,\mathrm{Im}(\langle v_2, v_3\rangle)$$

$$= \langle v_1, v_3\rangle + \langle v_2, v_3\rangle.$$

(2f) Let $(V, \|\,\|)$ be a normed vector space that satisfies

$$\text{if } x, y \in V \text{ then } \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Define $\langle\ \rangle : V \times V \to \mathbb{R}$ by

$$\langle x, y\rangle = \tfrac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$$

Show that $(V, \langle\ \rangle)$ is a positive definite inner product space.

To show: (a) If $v_1, v_2 \in V$ then $\langle v_2, v_1\rangle = \langle v_1, v_2\rangle$

(b) If $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2, v_3 \in V$ then

$$\langle c_1 v_1 + c_2 v_2, v_3\rangle = c_1\langle v_1, v_3\rangle + c_2\langle v_2, v_3\rangle.$$

(c) If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$.

(d) If $v \in V$ then $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$

(d) To show: If $v \in V$ then $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$

Assume $v \in V$.

To show: $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$

$$\begin{aligned}
\langle v, v\rangle &= \tfrac{1}{2}\left(\|v+v\|^2 - \|v\|^2 - \|v\|^2\right) \\
&= \tfrac{1}{2}\left(\|2v\|^2 - 2\|v\|^2\right) \\
&= \tfrac{1}{2}\left(4\|v\|^2 - 2\|v\|^2\right) \\
&= \|v\|^2 \in \mathbb{R}_{\geq 0}.
\end{aligned}$$

(c) To show: If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$.

Assume $v \in V$ and $\langle v, v\rangle = 0$.

To show: $v = 0$.

We know: $\|v\|^2 = \langle v, v\rangle = 0$

So $\|v\| = 0$

So $v = 0$.

(a) To show: If $v_1, v_2 \in V$ then $\langle v_2, v_1\rangle = \langle v_1, v_2\rangle$.

Assume $v_1, v_2 \in V$.

To show: $\langle v_2, v_1\rangle = \langle v_1, v_2\rangle$.

$$\begin{aligned}\langle v_2, v_1\rangle &= \tfrac{1}{2}\left(\|v_2+v_1\|^2 - \|v_2\|^2 - \|v_1\|^2\right)\\ &= \tfrac{1}{2}\left(\|v_1+v_2\|^2 - \|v_1\|^2 - \|v_2\|^2\right)\\ &= \langle v_1, v_2\rangle.\end{aligned}$$

(b) To show: If $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2, v_3 \in V$ then

$$\langle c_1 v_1 + c_2 v_2, v_3\rangle = c_1\langle v_1, v_3\rangle + c_2\langle v_2, v_3\rangle.$$

To show: (ba) If $v_1, v_2, v_3 \in V$ then

$$\langle v_1 + v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$$

(bb) If $c \in \mathbb{R}$ and $v_1, v_2 \in V$ then

$$\langle c v_1, v_2\rangle = c\langle v_1, v_2\rangle.$$

(ba) Assume $v_1, v_2, v_3 \in V$.

To show: $\langle v_1+v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$.

Adding

$$\frac{1}{2}\|v_1+v_3+v_2\|^2 + \frac{1}{2}\|v_1+v_3-v_2\|^2 = \|v_1+v_3\|^2 + \|v_2\|^2 \quad \text{and}$$

$$\frac{1}{2}\|v_2+v_3+v_1\|^2 + \frac{1}{2}\|v_2+v_3-v_1\|^2 = \|v_2+v_3\|^2 + \|v_1\|^2$$

gives

$$\|v_1+v_2+v_3\|^2 = \|v_1\|^2 + \|v_2\|^2 + \|v_1+v_3\|^2 + \|v_2+v_3\|^2 - \frac{1}{2}\|v_1+v_3-v_2\|^2 - \frac{1}{2}\|v_2+v_3-v_1\|^2$$

$$= \|v_1\|^2 + \|v_2\|^2 + \|v_1+v_3\|^2 + \|v_2+v_3\|^2 - \frac{1}{2}\left(\|v_3+(v_1-v_2)\|^2 + \|v_3-(v_1-v_2)\|^2\right)$$

$$= \|v_1\|^2 + \|v_2\|^2 + \|v_1+v_3\|^2 + \|v_2+v_3\|^2 - \|v_3\|^2 - \|v_1-v_2\|^2.$$

Using this

$$\langle v_1+v_2, v_3\rangle = \frac{1}{2}\left(\|v_1+v_3+v_2\|^2 - \|v_1+v_2\|^2 - \|v_3\|^2\right)$$

$$= \frac{1}{2}\left(\|v_1\|^2 + \|v_2\|^2 + \|v_1+v_3\|^2 + \|v_2+v_3\|^2 - \|v_3\|^2 - \|v_1-v_2\|^2 - \|v_1+v_2\|^2 - \|v_3\|^2\right)$$

$$= \frac{1}{2}\left(\|v_1+v_3\|^2 - \|v_1\|^2 - \|v_3\|^2 + \|v_2+v_3\|^2 - \|v_2\|^2 - \|v_3\|^2 + 2\|v_1\|^2 + 2\|v_2\|^2 - \left(\|v_1+v_2\|^2 + \|v_1-v_2\|^2\right)\right)$$

$$= \langle v_1, v_3\rangle + \langle v_2, v_3\rangle + \frac{1}{2}\left(2\|v_1\|^2 + 2\|v_2\|^2 - \left(\|v_1+v_2\|^2 + \|v_1-v_2\|^2\right)\right)$$

$$= \langle v_1, v_3\rangle + \langle v_2, v_3\rangle + 0 = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle.$$