(1) (a) Let (X,d) be a metric space.

Let $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$. The ball of radius ε at x is

$$B_\varepsilon(x) = \{y \in X \mid d(y,x) < \varepsilon\}.$$

The metric space topology on (X,d) is the collection

$$\mathcal{T} = \{\text{unions of } B_\varepsilon(x)\}_\emptyset$$

(b) A topological space $(X,\mathcal{T})$ is Hausdorff if $(X,\mathcal{T})$ satisfies

if $x,y \in X$ and $x \neq y$ then there exist open sets U and V in X with

$$x \in U, \quad y \in V \quad \text{and} \quad U \cap V = \emptyset.$$

To show: If (X,d) is a metric space and $\mathcal{T}$ is the metric space topology then $(X,\mathcal{T})$ is a Hausdorff topological space.

Assume: (X,d) is a metric space and $\mathcal{T}$ is the metric space topology.

To show: $(X,\mathcal{T})$ is Hausdorff.

To show: If $x,y \in X$ and $x \neq y$ then there exist open sets U and V in X with

$$x \in U, \quad y \in V \quad \text{and} \quad U \cap V = \emptyset.$$

Assume $x, y \in X$ and $x \neq y$.

To show: There exist open sets U and V in X with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Let

$U = B_{d/3}(x)$ and $V = B_{d/3}(y)$ where $d = d(x,y)$.

To show: (ba) U is open and V is open in X

(bb) $x \in U$ and $y \in V$.

(bc) $U \cap V = \emptyset$.

(ba) U and V are open in X since they are open balls.

(bb) $x \in U = B_{d/3}(x)$ since $d(x,x) = 0 < d/3$

(Note that $d \neq 0$ since $x \neq y$).

$y \in V = B_{d/3}(x)$ since $d(y,y) = 0 < d/3$.

(bc) To show: $U \cap V = \emptyset$.

Assume $z \in U \cap V$.

Then $d(z,x) < d/3$ and $d(z,y) < d/3$.

So $d = d(x,y) \leq d(x,z) + d(z,y) < d/3 + d/3 = \frac{2d}{3}$,

which is a contradiction.

So $U \cap V = \emptyset$.

So $(X, \mathcal{T})$ is Hausdorff.

(1c) A topological space $(X,\mathcal{T})$ is <u>normal</u> if $(X,\mathcal{T})$ satisfies:

if A and B are closed subsets of X and $A\cap B=\emptyset$ then there exist open sets U and V in X such that $A\subseteq U$ and $B\subseteq V$ and $U\cap V=\emptyset$.

Let (X,d) be a metric space and let

$$\mathcal{T}=\{\text{unions of open balls } B_\varepsilon(x)\}$$

be the metric space topology on X.

To show: $(X,\mathcal{T})$ is normal.

To show: If A and B are closed subsets of X and $A\cap B=\emptyset$ then there exist open sets U and V in X such that $A\subseteq U$ and $B\subseteq V$ and $U\cap V=\emptyset$.

Assume A and B are closed subsets of X and $A\cap B=\emptyset$.

To show: There exist open sets U and V in X such that $A\subseteq U$ and $B\subseteq V$ and $U\cap V=\emptyset$.

Let
$$U=\{x\in X \mid d(x,A)<d(x,B)\} \text{ and}$$
$$V=\{x\in X \mid d(x,B)<d(x,A)\}$$

where $d(x,A)=\inf\{d(x,a) \mid a\in A\}$ and
$d(x,B)=\inf\{d(x,b) \mid b\in B\}$.

To show: (a) U is open

(b) V is open

(c) $A \subseteq U$

(d) $B \subseteq V$

(e) $U \cap V = \phi$.

(e) If $x \in X$ and $d(x,A) < d(x,B)$ ~~and~~ then $d(x,B) \nless d(x,A)$.

So $U \cap V = \phi$.

(c) To show: $A \subseteq U$.

To show: If $a \in A$ then $a \in U$.

Assume $a \in A$.

To show: $a \in \{x \in X \mid d(x,A) < d(x,B)\}$.

To show: $d(a,A) < d(a,B)$.

Since $d(a,a) = 0$ and $d(a,A) \le d(a,a)$ then $d(a,A) = 0$.

To show: $0 < d(a,B)$.

To show: $\inf\{d(a,b) \mid b \in B\} > 0$.

If there exists $b_1, b_2, \ldots$ in B such that

$\lim_{n\to\infty} d(a,b_n) = 0$ then $\lim_{n\to\infty} b_n = a$ and $a \in \overline{B}$.

This is a contradiction to $A \cap B = \phi$.

So $\inf\{d(a,b) \mid b \in B\} \ne 0$.

So $A \subseteq U$.

(d) The proof of (d) is similar to the proof (c) with V replacing U, with A and B switched, with $b \in B$ instead of $a \in A$, and with $b_1, b_2, \ldots \in B$ replaced by $a_1, a_2, \ldots \in A$.

(a) To show: U is open.

To show: If $x \in U$ then x is an interior point of U.

Assume $x \in U$. Then $d(x, A) < d(x, B)$.

To show: x is an interior point of U.

To show: There exists $\varepsilon \in \mathbb{R}_{>0}$ such that $x \in B_\varepsilon(x) \subseteq U$.

Let $\varepsilon = \frac{d(x, B)}{2}$.

To show: $B_\varepsilon(x) \subseteq U$.

To show: If $y \in B_\varepsilon(x)$ then $d(y, A) < d(y, B)$.

Assume $y \in B_\varepsilon(x)$. Then $d(y, x) < \varepsilon$.

To show: $d(y, A) < d(y, B)$.

$$d(y, A) \le d(y, x) + d(x, A) < d(y, x) + d(x, B) < d(y, B).$$

So $B_\varepsilon(x) \subseteq U$.

So x is an interior point of U.

So U is open.

(b) The proof of (b) is similar with V replacing U.

(1d)

A topological space is first countable if $(X, \mathcal{T}_X)$ satisfies: if $a \in X$ then there exist $N_1, N_2, \ldots \in \mathcal{N}(a)$ such that if $N \in \mathcal{N}$ then there exists $j \in \mathbb{Z}_{>0}$ such that $N \supseteq N_j$.

Let (X, d_X) be a metric space.

To show: If $a \in X$ then there exist $N_1, N_2, \ldots \in \mathcal{N}(a)$ such that if $N \in \mathcal{N}(a)$ then there exists $j \in \mathbb{Z}_{>0}$ such that $N \supseteq N_j$.

Assume $a \in X$.

Let $N_1 = B_1(a)$, $N_2 = B_{\frac{1}{2}}(a)$, $N_3 = B_{\frac{1}{3}}(a)$, ...

To show: If $N \in \mathcal{N}(a)$ then there exists $j \in \mathbb{Z}_{>0}$ such that $N \supseteq N_j$.

Assume $N \in \mathcal{N}(a)$.

Then there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq N$.

Since U is a union of open balls there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $a \in B_\varepsilon(a) \subseteq U \subseteq N$

Let $j \in \mathbb{Z}_{>0}$ such that $\frac{1}{j} < \varepsilon$. Then

$$a \in B_{\frac{1}{j}}(a) \subseteq B_\varepsilon(a) \subseteq U \subseteq N.$$

So (X, d_X) is first countable.

(1e) Give an example of a topological space $(Y, \mathcal{U})$ which is not Hausdorff.

Let $Y = \{0,1\}$ and $\mathcal{U} = \{\phi, \{0\}, \{0,1\}\}$

Let $x = 0$ and $y = 1$. Then $x \neq y$.

If V is open and $1 = y \in V$ then $V = \{0,1\}$

If U is open and $0 = x \in U$ then $0 \in U \cap V$.

So, if U is open and $x \in U$ and V is open and $y \in V$ then $U \cap V \neq \phi$.

So $(Y, \mathcal{U})$ is not Hausdorff.

(1f) Give an example of a topological space $(Y, \mathcal{U})$ which is not normal.

Let $Y = \{0,1,2\}$ and $\mathcal{U} = \{\phi, \{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$.

Then $\mathcal{U}$ is a topology on Y and the closed sets are $\phi, \{1\}, \{2\}, \{1,2\}, \{0,1,2\}$.

Let $A = \{1\}$ and $B = \{2\}$ so that A and B are closed and $A \cap B = \phi$.

If $U \supseteq A$ and U is open then U is $\{0,1\}$ or $\{0,1,2\}$.

If $V \supseteq B$ and V is open then V is $\{0,2\}$ or $\{0,1,2\}$.

So, if U and V are open and $U \supseteq A$ and $V \supseteq B$ then $U \cap V \neq \phi$ (since $0 \in U \cap V$).

So $(Y, \mathcal{U})$ is not normal.

(1g) Let $X = \mathbb{R}$ with the topology

$$\mathcal{T} = \{ U \subseteq \mathbb{R} \mid U^c \text{ is finite} \}.$$

To show: $(X, \mathcal{T})$ is not first countable.

To show: There exists $a \in \mathbb{R}$ such that there does not exist $N_1, N_2, \ldots \in \mathcal{N}(a)$ such that if $N \in \mathcal{N}(a)$ then there exists $j \in \mathbb{Z}_{>0}$ such that $N \supseteq N_j$.

Let $a = 0$.

To show: There does not exist $N_1, N_2, \ldots \in \mathcal{N}(a)$ such that if $N \in \mathcal{N}(a)$ then there exists $j \in \mathbb{Z}_{>0}$ such that $N \supseteq N_j$.

Assume $N_1, N_2, \ldots \in \mathcal{N}(a)$.

To show: There exists $N \in \mathcal{N}(a)$ such that there does not exist $j \in \mathbb{Z}_{>0}$ such that $N \supseteq N_j$.

Since N_i^c is finite ~~there~~ $N_1^c \cup N_2^c \cup \cdots$ is countable.

Let $\gamma \in \mathbb{R}$ such that $\gamma \notin N_1^c \cup N_2^c \cup \cdots$ and $\gamma \neq 0$.

Let $N = \{\gamma\}^c$.

Then $N \in \mathcal{N}(0)$

Since $\gamma \notin N_1^c \cup N_2^c \cup \cdots$

then $\gamma \notin N_j^c$ for $j \in \mathbb{Z}_{>0}$.

So $N^c \not\subseteq N_j^c$ for $j \in \mathbb{Z}_{>0}$.

So $N \not\supseteq N_j$ for $j \in \mathbb{Z}_{>0}$.

So $(X, \mathcal{T})$ is not first countable.