The KP, KdV and NLS heirarchies

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The Kadomtsev-Petviashvili equation (KP equation) is

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right), \quad \text{for } u = u(x,y,t),$$

(in Hirota form: $(D_1^4 + 3D_2^2 - 4D_1D_3)\tau\cdot\tau = 0$).

The Korteweg-deVries equation (KdV equation) is

$$\frac{\partial u}{\partial t} = \frac{3}{2}u\frac{\partial u}{\partial x} + \frac{1}{4}\frac{\partial^3 u}{\partial x^3}, \quad \text{for } u = u(x,t),$$

(in Hirota form: $(D_1^4 - 4D_1D_3)\tau\cdot\tau = 0$).

The NLS system (nonlinear Schrödinger system) is

$$\begin{aligned} q_t &= q_{xx} - 2q^2q^*, \\ q^*_t &= -q^*_{xx} - 2q\,q^{*2}, \end{aligned} \quad \text{for } q = q(x,t) \text{ and } q^* = q^*(x,t),$$

and writing/forcing $g(x,t) = q^*(x,it) = \pm\overline{q(x,it)}$ gives the classical nonlinear Schrödinger equation is

$$ig_t = -g_{xx} \pm 2|g|^2 g, \quad \text{for } g = g(x,t).$$

POINT:

(a) The bilinear identity for

GL_∞ on $B^{(0)} \cong L(\Lambda_0)$ in the principal vertex op. const.

produces the KP hierarchy, in which the

coeff. of y_3 produces the KP equation

(b) The bilinear identity for

$\widehat{SL}_2$ on $L(\Lambda_0)$ in the princ. vertex operator const.

produces the KdV hierarchy, in which the

coeff. of $y_1 y_2$ produces the KdV equation

(c) The bilinear identity for

$\widehat{SL}_2$ on $L(\Lambda_0)$ in the homogeneous vertex op. construction

produces the NLS hierarchy, in which the

coeff. of y_1^2 produces the NLS system.

<u>The orbit $G\cdot v^+$ and the bilinear identity</u>

$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ is the affine Lie algebra.

The irreducible integrable $\mathfrak{g}$-modules in $\mathcal{O}$ are indexed by

$$\mathfrak{h}^*_{\mathbb{Z}_{\geq 0}} = \mathbb{Z}_{\geq 0}\text{-span}\{\Lambda_0, \Lambda_1, \ldots, \Lambda_\ell\} = \mathbb{Z}_{\geq 0}\Lambda_0 + \cdots + \mathbb{Z}_{\geq 0}\Lambda_\ell$$

where $\Lambda_0, \Lambda_1, \ldots, \Lambda_\ell$ are the <u>fundamental</u> <u>weights</u> for $\mathfrak{g}$

Denote the irred. int. $\mathfrak{g}$-module of highest weight Λ by

$L(\Lambda) = \mathcal{U}v^+$, where $\mathcal{U} = \mathcal{U}\mathfrak{g}$ is the env. alg. of $\mathfrak{g}$.

Let $\mathring{\mathfrak{h}}$ be a Cartan subalgebra of $\mathring{\mathfrak{g}}$, and

$\mathfrak{h} = \mathring{\mathfrak{h}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$, and write

$$\mathfrak{g} = \Big(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha\Big) \oplus \mathfrak{h}, \text{ as } \mathfrak{h}\text{-modules.}$$

Let

$$\Omega_2 = \sum_{\alpha \in R} \sum_{i=1}^{d_\alpha} e_\alpha^{(i)} \otimes e_\alpha^{(i)}$$

where $d_\alpha = \dim(\mathfrak{g}_\alpha)$ and $\{e_\alpha^{(1)}, \ldots, e_\alpha^{(d_\alpha)}\}$ is a basis of $\mathfrak{g}_\alpha$.

<u>Kac Prop. 14.12</u> Let $G = \exp(\mathfrak{g})$ so that $\mathfrak{g} = \mathrm{Lie}(G)$. Then

$$G\cdot v^+ \subseteq \{\tau \in L(\Lambda) \mid \Omega_2(\tau \otimes \tau) = (\Lambda|\Lambda)\cdot(\tau \otimes \tau)\}$$

where $\Omega_2(\tau\otimes\tau) \in L(\Lambda) \otimes L(\Lambda)$.

$\Omega_2(\tau\otimes\tau) = (\Lambda|\Lambda)(\tau\otimes\tau)$ is the <u>bilinear identity</u>

tau-functions, n-solitons, Hirota polynomials, Hirota eqns

The principal vertex operator construction of $L(\Lambda_0)$ has

$$L(\Lambda_0) = \mathbb{C}[x_j \mid j \in E_+] \quad \text{with } v^+ = 1.$$

A τ-function is an element $\tau \in \widehat{G \cdot 1}$,

$$\widehat{G \cdot 1} \subseteq \mathbb{C}[[x_j \mid j \in E_+]] = \widehat{L(\Lambda_0)},$$

where $\widehat{\ }$ denotes completion and $G = \exp(\mathfrak{g})$.

An n-soliton is

$$\tau = e^{c_1 X_{\beta_1}(k_1)} \cdots e^{c_n X_{\beta_n}(k_n)} \cdot 1, \quad \text{with } \begin{matrix} c_1, \dots, c_n \in \mathbb{C}, \; k_1, \dots, k_n \in \mathbb{C} \\ X_{\beta_1}, \dots, X_{\beta_n} \in \mathring{\mathfrak{g}}, \end{matrix}$$

where $X_\beta(z) = \sum_{j \in \mathbb{Z}} X_\beta(n) z^{-n-1}$, where $X_\beta(n) = X_\beta t^n \in \mathring{\mathfrak{g}}[t, t^{-1}]$.

Write

$$L(\Lambda_0) \otimes L(\Lambda_0) = \mathbb{C}[x_j', x_j'' \mid j \in E_+] \quad \text{and let} \quad \begin{matrix} x_j = \tfrac{1}{2}(x_j' + x_j'') \\ y_j = \tfrac{1}{2}(x_j' - x_j'') \end{matrix}$$

The set of Hirota polynomials is

$$Hir = \mathbb{C}[y_j \mid j \in E_+] \cap L(2\Lambda_0)^\perp$$

a subset of $L(\Lambda_0) \otimes L(\Lambda_0) = L(2\Lambda_0) \oplus L(2\Lambda_0)^\perp$.

Hirota notation:

$$P(D_1, D_2, \dots) f \cdot g = P\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots\right) f(x_1+u_1, x_2+u_2, \dots)\, g(x_1-u_1, x_2-u_2, \dots) \Big|_{u_i=0}$$

Theorem

$$\widehat{G \cdot 1} \subseteq \{ \tau \in \widehat{L(\Lambda_0)} \mid \underbrace{P\left(\tfrac{1}{2} D_1, \tfrac{1}{2}\tfrac{D_2}{2}, \tfrac{1}{2}\tfrac{D_3}{3}, \dots\right) \tau \cdot \tau = 0}_{\text{Hirota equation}} \text{ for } P \in Hir \}$$

τ-functions and the loop Grassmannian

$$\begin{array}{lcl}
\hat{\mathfrak{g}} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d & & \hat{G} \\
\cup\!\!| & & \cup\!\!| \\
\hat{\mathfrak{g}}' = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d & & \hat{G}' \\
\cup\!\!| & \text{is the} & \cup\!\!| \\
\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] & \text{Lie algebra} & G = \mathring{G}(\mathbb{C}((t))) \\
\cup\!\!| & \text{of} & \cup\!\!| \\
\mathring{\mathfrak{g}} & & \mathring{G}
\end{array}$$

$$\begin{array}{l}
G = \mathring{G}(\mathbb{C}((t))) \\
\cup\!\!| \\
K = \mathring{G}(\mathbb{C}[[t]]),
\end{array}$$

G/K is the <u>loop</u> <u>Grassmannian</u>

Then

$$L(\Lambda_0) = H^0(G/K, \mathcal{L}_{\Lambda_0})$$

Borel-Weil-Bott
Kumar-Mathieu

and

$$\begin{array}{ccc} G/K & \hookrightarrow & \mathbb{P}(L(\Lambda_0)) \\ gK & \longmapsto & gv^+ \end{array}$$

with image $G \cdot v^+$.

So, a <u>τ-function</u> is an element of $Gv^+ = G/K$.

The Hirota equations, or bilinear identity, or KP hierarchy, are the

<u>Plücker relations</u> for G/K in $\mathbb{P}(L(\Lambda_0))$

i.e. the equations describing

G/K as a subvariety of $\mathbb{P}(L(\Lambda_0))$.

The principal vertex operator construction of $L(\Lambda_0)$

Denote Kac-Moody generators of

$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ by $e_0, e_1, \ldots, e_\ell, f_0, f_1, \ldots, f_\ell$.

Let $e = e_0 + e_1 + \cdots + e_\ell$ and $\mathfrak{s} = Z_{\mathfrak{g}}(e)$.

Kac Lemma 14.4, Prop. 14.4, (14.6.1)–(14.6.5), Lemma 14.6:

(a) $\mathfrak{s}$ has a basis $\{K, p_1, p_2, \ldots, q_1, q_2, \ldots\}$ with

$[p_i; p_j] = 0, \quad [q_i; q_j] = 0, \quad K \in Z(\mathfrak{s}), \quad [p_i; q_j] = \delta_{ij} K.$

(b) $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{s}}(L(\Lambda_0))$ is an irreducible $\mathfrak{s}$-module

(c) Realize $L(\Lambda_0) = \mathbb{C}[x_1, x_2, \ldots]$ with

p_i acting as $\frac{\partial}{\partial x_i}$, q_i acting as x_i, K acting as $\mathrm{Id}_{L(\Lambda_0)}$.

For $X_\beta \in \mathring{\mathfrak{g}}_\beta$ let

$$X_\beta(z) = \sum_{n \in \mathbb{Z}} X_\beta(n) z^{-n-1}, \quad \text{where } X_\beta(n) = X_\beta t^n \in \mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]$$

(root vectors: $\mathring{\mathfrak{g}} = \mathring{\mathfrak{h}} \oplus \left(\bigoplus_{\beta \in \mathring{R}} \mathring{\mathfrak{g}}_\beta\right)$ as $\mathring{\mathfrak{h}}$-modules).

Then $X_\beta(z)$ acts on $L(\Lambda_0) = \mathbb{C}[[x_1, x_2, \ldots]]$ by

$$\Gamma^\beta(z) = \langle \Lambda_0, X_{\beta,0} \rangle \left(\exp\left(\sum_{j \in \mathbb{Z}_{>0}} \lambda_{\beta,j} z^{b_j} x_j \right) \exp\left(-\sum_{j \in \mathbb{Z}_{>0}} \lambda_{\beta, N+1-j} z^{-b_j} \frac{\partial}{\partial x_j} \right) \right)$$

where $\lambda_{\beta,j} = \ldots$, $X_{\beta,0} = \ldots$, $b_j = \ldots$.

(the b_j are the "positive exponents" of $\mathfrak{g}$).

Conversion: Hirota notation to differential equations

$$e^{y_1 D_1 + y_2 D_2 + \cdots} f\cdot g = f(x_1+y_1, x_2+y_2, \ldots)\, g(x_1-y_1, x_2-y_2, \ldots).$$

Use the definition of e^z to expand the left hand side as

$$\begin{aligned} e^{y_1 D_1 + y_2 D_2 + \cdots} f\cdot g = {} & f\cdot g + (D_1 f\cdot g) y_1 + (D_2 f\cdot g) y_2 + \cdots \\ & + \tfrac{1}{2}(D_1^2 f\cdot g) y_1^2 + \tfrac{1}{2}(2D_1 D_2 f\cdot g) y_1 y_2 + \tfrac{1}{2}(D_2^2 f\cdot g) y_2^2 + \cdots \\ & + \tfrac{1}{3!}(D_1^3 f\cdot g) y_1^3 + \tfrac{1}{3!}(3D_1^2 D_2 f\cdot g) y_1^2 y_2 + \cdots \end{aligned}$$

and use Taylor's formula to expand the right hand side as

$$f(x_1+y_1, x_2+y_2, \ldots)\, g(x_1-y_1, x_2-y_2, \ldots)$$

$$= \left(\sum_{k_1, k_2, \ldots} \left(\frac{1}{k_1!}\frac{1}{k_2!}\cdots \left(\frac{\partial}{\partial x_1}\right)^{k_1} \left(\frac{\partial}{\partial x_2}\right)^{k_2} \cdots f \right) y_1^{k_1} y_2^{k_2} \cdots \right) \left(\sum_{l_1, l_2, \ldots} \left(\frac{1}{l_1!}\frac{1}{l_2!}\cdots \left(\frac{\partial}{\partial x_1}\right)^{l_1} \left(\frac{\partial}{\partial x_2}\right)^{l_2} \cdots g \right) (-y_1)^{l_1} (-y_2)^{l_2} \cdots \right)$$

Comparing coefficients of $y_1^{s_1} y_2^{s_2}\cdots$ on each side gives

$$D_1 f\cdot g = \frac{\partial f}{\partial x_1} g - f \frac{\partial g}{\partial x_1},$$

$$D_1^2 f\cdot g = \frac{\partial^2 f}{\partial x_1^2} g - 2\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + f\frac{\partial^2 g}{\partial x_1^2},$$

$$D_1 D_3 f\cdot g = \frac{\partial^2 f}{\partial x_1 \partial x_3} g - \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_3} - \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_1} + f\frac{\partial^2 g}{\partial x_1 \partial x_3},$$

$$D_1^4 f\cdot g = \frac{\partial^4 f}{\partial x_1^4} g - 4\frac{\partial^3 f}{\partial x_1^3}\frac{\partial g}{\partial x_1} + \frac{4!}{2!2!}\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 g}{\partial x_1^2} - \frac{4!}{3!}\frac{\partial f}{\partial x_1}\frac{\partial^3 g}{\partial x_1^3} + f\frac{\partial^4 g}{\partial x_1^4},$$

etc.