**MAST30026 Metric and Hilbert spaces**

**Assignment 1 (10 questions)**

**due Wednesday Sep 10 at 10am**

**1.** Let $A$ and $B$ be bounded subsets of a metric space $(X, d)$ such that $A \cap B \neq \emptyset$. Show that

$$\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B).$$

What can you say if $A$ and $B$ are disjoint?

**2.** Let $X = C[0,1] = \{f \colon [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$. The supremum metric $d_\infty \colon X \times X \to \mathbb{R}_{\geq 0}$ and the $L^1$ metric $d_1 \colon X \times X \to \mathbb{R}_{\geq 0}$ are defined by

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\} \qquad \text{and}$$
$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Consider the sequence $\{f_1, f_2, f_3, \ldots\}$ in $X$ where $f_n(x) = nx^n(1-x)$ for $0 \leq x \leq 1$.

(a) Determine whether $\{f_n\}$ converges in $(X, d_1)$.
(b) Determine whether $\{f_n\}$ converges in $(X, d_\infty)$.

**3.** Let $X$ and $Y$ be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$. Show that

$$\overline{A} \times \overline{B} = \overline{A \times B}.$$

**4.** Let $(X, d)$ be a metric space and let $A$ be a non-empty subset of $X$. Recall that for each $x \in X$, the distance from $x$ to $A$ is

$$d(x, A) = \inf\{d(x,a) \mid a \in A\}.$$

(a) Prove that $\overline{A} = \{x \in X \mid d(x, A) = 0\}$.
(b) Prove that $|d(x,A) - d(y,A)| \leq d(x,y)$ for all $x, y \in X$. [Hint: first show that $d(x,A) \leq d(x,y) + d(y,A)$.]
(c) Deduce the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.
(d) Show that if $x \notin \overline{A}$ then $U = \{y \in X \mid d(y,A) < d(x,A)\}$ is an open set in $X$ such that $\overline{A} \subseteq U$ and $x \notin U$.

**5.** Determine whether the following sequences of functions converge uniformly.

(a) $f_n(x) = e^{-nx^2}, \quad x \in [0,1]$;
(b) $g_n(x) = e^{-x^2/n}, \quad x \in [0,1]$.
(c) $g_n(x) = e^{-x^2/n}, \quad x \in \mathbb{R}$.

**6.** Let $X$ be the set of all real sequences with *finitely many non-zero terms* with the supremum metric: if $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ then $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i| \mid i \in \mathbb{Z}_{>0}\}$.
For each $n \in \mathbb{N}$, let $\mathbf{x}^n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$.

(a) Show that $\{\mathbf{x}^n\}$ is a Cauchy sequence in $X$.
(b) Show that $\{\mathbf{x}^n\}$ does not converge to a point in $X$. (So $X$ is not complete.)

**7.** Let $X$ be a nonempty set and let $(Y, d)$ be a complete metric space. Let $f : X \to Y$ be an injective function and define

$$d_f(x, y) = d(f(x), f(y)), \qquad \text{for } x, y \in X.$$

(a) Show that $d_f$ is a metric on $X$.
(b) Show that $(X, d_f)$ is a complete metric space if $f(X)$ is a closed subset of $Y$.

**8.** Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given by

$$f(x) = \frac{2}{2 + x}.$$

(a) Show that $f$ defines a contraction mapping $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.
(b) Fix $x_0 \geq 0$ and $x_{n+1} = f(x_n)$ for all $n \geq 0$. Show that the sequence $\{x_n\}$ converges and find its limit with respect to the usual metric on $\mathbb{R}$.

**9.** Let $X$ be a connected topological space. Let $f\colon X \to \mathbb{R}$ be continuous with $f(X) \subseteq \mathbb{Q}$. Show that $f$ is a constant function.

**10.** Show that $X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ is not homeomorphic to $\mathbb{R}$.