Lecture 9: Metric and Hilbert spaces 12 August 2014

Convergence

Let (X,d) be a metric space and let $x \in X$.

First definition: A function $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ **converges** to x if $\vec{x}$ satisfies:

> if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(x_n, x) < \varepsilon$.

Write $\lim_{n\to\infty} x_n = x$ if $\vec{x}$ converges to x.

Second definition: A function $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ **converges** to x if $\lim_{n\to\infty} d(x_n, x) = 0$.

HW: Let (X,d) be a metric space and let $x \in X$. Let $\vec{x}: \mathbb{Z}_{>0} \to X$ be a function. Show that $\vec{x}$ satisfies $(*)$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

Uniqueness of limits

HW: Let $\vec{x}: \mathbb{Z}_{>0} \to X$ and let $x, y \in X$. Show that if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$ then $x = y$.

Equivalent metrics

Let X be a set and let $d_1: X\times X \to \mathbb{R}_{\geq 0}$ and $d_2: X\times X \to \mathbb{R}_{\geq 0}$ be metrics on X. The metrics d_1 and d_2 are <u>equivalent</u> if d_1 and d_2 satisfy:

(*) If $\vec{x}: \mathbb{Z}_{\geq 0} \to X$, $n \mapsto x_n$ and $x \in X$ then

$$\lim_{n\to\infty} d_1(x_n, x) = 0 \text{ if and only if } \lim_{n\to\infty} d_2(x_n, x) = 0.$$

<u>HW</u>: Show that if d_1 and d_2 satisfy

if $x, y \in X$ then there exist $C_1 \in \mathbb{R}_{>0}$ and $C_2 \in \mathbb{R}_{>0}$ such that

$$d_1(x,y) \leq C_1 d_2(x,y) \quad\text{and}\quad d_2(x,y) \leq C_2 d_1(x,y)$$

then d_1 and d_2 satisfy (*).

<u>HW</u>: (a) Is the "if $x, y \in X$" in the right place or should it be after "such that"?

(b) Why isn't this statement if and only if?

Convergence and closure

Let (X,d) be a metric space.

Then X is a metric space with the metric space topology.

Let $A \subseteq X$ and let $\bar{A}$ be the closure of A.

HW Show that

$$\bar{A} = \{x \in X \mid \text{there exists } \vec{a}: \begin{matrix} \mathbb{Z}_{>0} \to A \\ n \mapsto a_n \end{matrix} \text{ such that } \lim_{n\to\infty} a_n = x\}$$

Proof: Let $R = \{x \in X \mid \text{there exists } \vec{a}: \mathbb{Z}_{>0} \to A \text{ with } \lim_{n\to\infty} a_n = x\}$.

To show: (a) $R \subseteq \bar{A}$

(b) $\bar{A} \subseteq R$.

(a) To show: If $x \in R$ then $x \in \bar{A}$.

Assume $x \in R$

To show: $x \in \bar{A}$

We know: there exists $\vec{a}: \begin{matrix} \mathbb{Z}_{>0} \to A \\ n \mapsto a_n \end{matrix}$ with $\lim_{n\to\infty} a_n = x$.

To show: x is a close point of A.

To show: If V is a neighborhood of x then there exists $a \in A$ such that $a \in V$.

Assume V is a neighborhood of x.

Then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B(x,\varepsilon) \subseteq V$.

To show: there exists $a \in A$ such that $a \in V$.

Let $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n \geq N$ then $d(a_n, x) < \varepsilon$.

Let $a = a_N$.

Then $d(a, x) = d(a_N, x) < \varepsilon$.

So $a \in B(x, \varepsilon) \subseteq V$.

So x is a close point of A.

So $R \subseteq \overline{A}$.

(b) Let $x \in \overline{A}$

To show: $x \in R$.

To show: There exists $\vec{a}: \begin{matrix} \mathbb{Z}_{>0} & \to & X \\ n & \mapsto & a_n \end{matrix}$ with $\lim_{n\to\infty} a_n = x$.

We know: x is a close point of A.

Let $n \in \mathbb{Z}_{>0}$ and let $a_n \in A$ such that $a_n \in B(x, \frac{1}{n})$.

Let $\vec{a}: \mathbb{Z}_{>0} \to A$ be given by $\vec{a}(n) = a_n$.

To show: $\lim_{n\to\infty} a_n = x$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, x) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $N \in \mathbb{Z}_{>0}$ be minimal such that $N > \frac{1}{\varepsilon}$.

To show: If $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, x) < \varepsilon$.

Assume $n \in \mathbb{Z}_{>0}$ and $n > N$.

To show: $d(a_n, x) < \varepsilon$.

Since $a_n \in B(x, \frac{1}{n})$,

$$d(a_n, x) < \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

So $\lim_{n \to \infty} a_n = x$.

So $x \in R$.

So $\bar{A} \subseteq R$. //.

Some definitions

Let X be a topological space. Let $A \subseteq X$.

The boundary of A is $\partial A = \bar{A} \cap \overline{(A^c)}$.

The set A is dense in X if $\bar{A} = X$.

The set A is nowhere dense in X if $(\bar{A})^\circ = \phi$.

Examples: (1) $\mathbb{Q}$ is dense in $\mathbb{R}$.

(2) $(0,1]$ is dense in $[0,1]$.

(3) The boundary of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$.

(4) The boundary of $(0,1]$ in $\mathbb{R}$ is $\{0,1\}$

since $\overline{(0,1]} \cap \overline{((0,1]^c)} = [0,1] \cap \overline{(-\infty,0] \cup (1,\infty)}$

$= [0,1] \cap ((-\infty,0] \cup [1,\infty)) = \{0,1\}$.

(5) $\mathbb{Z}_{>0}$ and $\mathbb{Z}$ are nowhere dense in $\mathbb{R}$.

(6) $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$.

(7) The Cantor set is nowhere dense in $[0,1]$.

The Cantor set is closed.