<u>Theorem</u> Let $J \subseteq \mathbb{R}$ with $J \neq \emptyset$. The subset J is connected if and only if J is an interval.

<u>Proof</u> $\Rightarrow$: Assume J is not an interval.

Let $x, y \in J$ and $z \in \mathbb{R}$ with

$$x < z < y, \qquad x, y \in J, \quad \text{and} \quad z \notin J.$$

Let $A = (-\infty, z) \cap J$ and $B = (z, \infty) \cap J$.

Then A and B are open subsets of J and

$$A \neq \emptyset, \quad B \neq \emptyset, \quad A \cap B = \emptyset \quad \text{and} \quad A \cup B = J.$$

So J is not connected.

$\Leftarrow$: Assume J is an interval.

To show: J is connected.

Proof by contradiction.

Assume J is not connected.

Let $A \subseteq J$ and $B \subseteq J$ be open subsets of J such that

$$A \cap B = \emptyset, \quad A \neq \emptyset, \quad B \neq \emptyset \quad \text{and} \quad A \cup B = J.$$

Then $f: J \longrightarrow \{0, 1\}$ given by

$$f(z) = \begin{cases} 0, & \text{if } z \in A, \\ 1, & \text{if } z \in B \end{cases}$$

is a continuous surjective function.

Let $x_1, y_1 \in J$ with $f(x_1)=0$ and $f(y_1)=1$.

Switching A and B if necessary we may assume that $x_1 < y_1$.

Construct sequences $x_1, x_2, \dots$ and $y_1, y_2, \dots$ by

$$x_{i+1} = \frac{x_i+y_i}{2} \text{ and } y_{i+1} = y_i, \quad \text{if } f\left(\frac{x_i+y_i}{2}\right)=0,$$

$$x_{i+1} = x_i \text{ and } y_{i+1} = \frac{x_i+y_i}{2}, \quad \text{if } f\left(\frac{x_i+y_i}{2}\right)=1.$$

By induction, $x_i \in J$ and $y_i \in J$, and, since J is an interval, $\frac{x_i+y_i}{2} \in J$ so that $f\left(\frac{x_i+y_i}{2}\right)$ is defined and

$$x_{i+1} \in J \text{ and } y_{i+1} \in J.$$

Also,

$f(x_{i+1})=0$, $f(y_{i+1})=1$, $x_i \le x_{i+1} < y_{i+1} \le y_i$ and

$|x_{i+1}-y_{i+1}| \le \frac{1}{2}|x_i-y_i|$ so that $|x_{i+1}-y_{i+1}| \le \frac{1}{2^i}|x_1-y_1|$.

Since $\mathbb{R}$ is complete and the sequence $x_1, x_2, \dots$ is increasing and bounded by y_1, $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$.

Since $\mathbb{R}$ is complete and the sequence $y_1, y_2, \dots$ is decreasing and bounded by x_1, $\lim_{n\to\infty} y_n$ exists in $\mathbb{R}$.

Since $\lim_{n\to\infty} |x_n - y_n| = 0$ then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

Let
$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

Since $x_1 \le x_2 \le \cdots \le x_n < y_n \le y_{n-1} \le \cdots \le y_1$ for $n \in \mathbb{Z}_{>0}$

then
$$x_1 < z < y_1.$$

Since J is an interval $z \in J$.

Since f is continuous,
$$0 = \lim_{n\to\infty} f(x_n) = f(z) = \lim_{n\to\infty} f(y_n) = 1.$$

This is a contradiction.

So J is connected. //

Notes and References

The proof of the theorem follows the proof given in the course notes of J. Hyam Rubinstein for "Metric and Hilbert spaces" at University of Melbourne. This proof does not differ substantially from the proof in [Bou, Gen Top. Ch. IV §2 No. 5 Theorem 4] but is organised to be more self contained.