lecture notes for Paul Sobaje for metric and Hilbert spaces 4-9 August 2014

Homework questions

(1) Let X be a set. Show that the discrete topology on X is a topology on X.

(2) Let (X, d) be a metric space. Show that the metric space topology on X is a topology on X.

(3) Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. Show that the subspace topology on Y is a topology on Y.

(4) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Show that the product topology on $X \times Y$ is a topology on $X \times Y$.

One could do these, one by one, directly, using proof machine. Alternatively, one can develop a few helpful definitions that, more or less, do them all in one fell swoop.

Let X be a topological space with topology $\mathcal{T}$.

A <u>base</u> of the topology on X is a collection $\mathcal{B} \subseteq \mathcal{T}$ such that

if $U \in \mathcal{T}$ then there exists $\mathcal{S} \subseteq \mathcal{B}$ such that

$$U = \bigcup_{A \in \mathcal{S}} A.$$

Let $x \in X$.

A <u>fundamental system</u> <u>of neighborhoods of x</u> is a set $\mathcal{S} \subseteq \mathcal{N}(x)$ such that

if $N \in \mathcal{N}(x)$ then there exists $W \in \mathcal{S}$ such that

$$W \subseteq N.$$

<u>HW1</u> Show that $\mathcal{B}$ is a base of $\mathcal{T}$ if and only if $\mathcal{B}$ satisfies

if $x \in X$ then $\{V \in \mathcal{B} \mid x \in V\}$ is a fundamental system of neighborhoods of x.

<u>HW2</u> Let (X, d) be a metric space. Show that the set of open balls is a base of the metric space topology on X by showing that it satisfies the condition in HW1.

HW3 Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological spaces. Show that

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{T}_x \text{ and } V \in \mathcal{T}_y \}$$

is a base of the product topology on $X \times Y$ by showing that it satisfies the condition in HW1.

Let $(X, \mathcal{T})$ be a topological space.

A connected set is a subset $E \subseteq X$ such that there do not exist open sets A and B $(A, B \in \mathcal{T})$ with

$$A \cap E \neq \emptyset \quad \text{and} \quad B \cap E \neq \emptyset \quad \text{and}$$

$$A \cup B \supseteq E \quad \text{and} \quad (A \cap B) \cap E = \emptyset .$$

Perhaps it is better to think of E with the subspace topology $\mathcal{T}_E$. Then E is connected if there do not exist U and V open in E $(U, V \in \mathcal{T}_E)$ such that

$$U \neq \emptyset \quad \text{and} \quad V \neq \emptyset \quad \text{and} \quad U \cup V = E$$

$$\text{and} \quad U \cap V = \emptyset .$$

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

Let $f: X \to Y$ be a function.

The function $f: X \to Y$ is <u>continuous</u> if f satisfies:

$$\text{if } V \in \mathcal{T}_Y \text{ then } f^{-1}(V) \in \mathcal{T}_X.$$

Recall that, by definition,

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}.$$

Recall that, by definition, a function $f: X \to Y$ is a subset $\Gamma \subseteq X \times Y$ such that

if $x \in X$ then there exists a unique $y \in Y$ such that $(x, y) \in \Gamma$.

Use the notation $f(x)$ so that

$$\Gamma = \{(x, f(x)) \mid x \in X\}$$

<u>Proposition</u> Let $f: X \to Y$ be a continuous function. Let $E \subseteq X$.

If E is connected

then $f(E)$ is connected.

Proof Assume E is connected.

To show: $f(E)$ is connected.

Proof by contradiction.

Assume $f(E)$ is not connected.

Let A and B be open in $f(E)$ such that

$$A \neq \emptyset \text{ and } B \neq \emptyset \text{ and } A \cup B \supseteq f(E) \text{ and } A \cap B = \emptyset$$

Let $C = f^{-1}(A)$ and $D = f^{-1}(B)$.

Then

$$C \cup D = f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) \supseteq f^{-1}(f(E)) \supseteq E$$

and

$$C \cap D = f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$$

and

$C \neq \emptyset$ since $A \neq \emptyset$ and $A \subseteq f(E)$,

$D \neq \emptyset$ since $B \neq \emptyset$ and $B \subseteq f(E)$.

So E is not connected. This is a contradiction

So $f(E)$ is connected //.