Lecture notes for Paul Sobaje for Metric and Hilbert Spaces 4-9 August 2014

Example Let $(X, \mathcal{T})$ be a topological space.

Let $\mathcal{C}$ be a collection of connected subsets of X such that

$$\bigcap_{A \in \mathcal{C}} A \neq \emptyset$$

Prove that $\bigcup_{A \in \mathcal{C}} A$ is connected

Proof Let $M = \bigcup_{A \in \mathcal{C}} A$ and let $x \in \bigcap_{A \in \mathcal{C}} A$.

Proof by contradiction. Assume M is not connected.

Let B and C be open sets such that

$$M \subseteq B \cup C, \quad M \cap B \cap C = \emptyset, \quad M \cap B \neq \emptyset \text{ and } M \cap C \neq \emptyset$$

Then $x \in B$ or $x \in C$. Assume $x \in B$.

Let $U \in \mathcal{C}$ be such that $C \cap U \neq \emptyset$.

Since $x \in \bigcap_{A \in \mathcal{C}} A$ then $x \in U$, so that

$$x \in B \cap U \text{ and } B \cap U \neq \emptyset.$$

Since $U \subseteq M$ then

$$U \subseteq B \cap C \text{ and } U \cap B \cap C = \emptyset.$$

This is a contradiction to U being connected.

So M is connected. $\triangle$

Example Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be connected. Show that $\bar{A}$ is connected.

Proof Proof by contradiction. Assume $\bar{A}$ is not connected. Let M and N be open subsets of $\bar{A}$ such that

$M \cup N = \bar{A}$, $M \neq \phi$, $N \neq \phi$ and $M \cap N = \phi$.

Then

$$(M \cap A) \cup (N \cap A) = A \quad \text{and} \quad (M \cap A) \cap (N \cap A) = \phi$$

There exists $x \in \bar{A} \cap M$, x is a close point of A and, since M is open, M is a neighborhood of x.

So $M \cap A \neq \phi$.

There exists $y \in \bar{A} \cap N$, ~~y~~ y is ~~a~~ close point of A and, since N is open, N is a neighborhood of y.

So $N \cap A \neq \phi$.

This is a contradiction to A is connected

So $\bar{A}$ is connected //.

Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$.
The <u>connected component</u> of x is

$$C_x = \bigcup_{\substack{A \text{ connected} \\ x \in A}} A,$$

the union of the connected subsets of X containing x.

<u>Proposition</u> (a) C_x is connected
(b) C_x is closed
(c) If $y \in C_x$ then $C_y = C_x$.
(d) If $x, y \in X$ then $C_x = C_y$ or $C_x \cap C_y = \emptyset$.

<u>Proof</u>
(a) This follows from Example 1.
(b) By Example 2, $\overline{C_x}$ is a connected set that contains x. So $\overline{C_x} \subseteq C_x$. So $\overline{C_x} = C_x$.
(c) Assume $y \in C_x$. Then C_x is a connected set containing y. So $C_x \subseteq C_y$.
Then C_y is a connected set containing x. So $C_y \subseteq C_x$.
So $C_x = C_y$.
(d) Assume $x, y \in X$ and $C_x \cap C_y \neq \emptyset$.
Let $z \in C_x \cap C_y$. So $z \in C_x$ and $z \in C_y$.
By (c), $C_x = C_z = C_y$. So $C_x = C_y$. //

<u>Theorem</u> Let $(X, \mathcal{T})$ be a topological space. Then X is connected if and only if there does not exist a continuous surjective function $f: X \to \{0,1\}$, where $\{0,1\}$ has the discrete topology.

<u>Proof</u> $\Rightarrow$ To show: If there exists a continuous surjective function $f: X \to \{0,1\}$ then X is not connected.

Assume that $f: X \to \{0,1\}$ is a continuous surjective function. Let

$$A = f^{-1}(0) \quad \text{and} \quad B = f^{-1}(1)$$

Since f is continuous, A and B are open.
Since f is surjective, $f^{-1}(0) = A \neq \emptyset$ and $f^{-1}(1) = B \neq \emptyset$.
Then

$$A \cup B = f^{-1}(\{0,1\}) = X \quad \text{and} \quad A \cap B = f^{-1}(0) \cap f^{-1}(1) = \emptyset.$$

So X is not connected.

$\Leftarrow$ To show: If X is not connected then there exists a continuous surjective function $f: X \to \{0,1\}$.

Assume X is not connected.

Then there exist open sets A and B such that

$$A \cup B = X, \quad A \neq \emptyset, \quad B \neq \emptyset \quad \text{and} \quad A \cap B = \emptyset.$$

Define $f: X \to \{0,1\}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \in B. \end{cases}$$

Since $X = A \cup B$ and $A \cap B = \emptyset$, f is well defined.

Since $A \neq \emptyset$ and $B \neq \emptyset$, f is surjective.

Since $A = f^{-1}(0)$ and $B = f^{-1}(1)$ are open, f is continuous.

So there exists a continuous surjective function $f: X \to \{0,1\}$. //.