A <u>topological</u> <u>space</u> is a set X with a collection $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U \in \mathcal{S}} U \right) \in \mathcal{T}$

(c) If $n \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_n \in \mathcal{T}$ then

$$U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}.$$

Let $(X, \mathcal{T})$ be a topological space.

An <u>open</u> <u>set</u> of X is $U \in \mathcal{T}$.

A <u>closed</u> <u>set</u> of X is a subset $E \subseteq X$ such that E^c is open.

<u>Examples</u>

(1) Let X be a set.

The <u>discrete</u> <u>topology</u> on X is

$$\mathcal{T} = \{\text{subsets of } X\}$$

(the "power set of X").

(2) Let (X,d) be a metric space.

Let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$.

The <u>ball of radius ε at x</u> is the set

$$B_\varepsilon(x) = \{ p \in X \mid d(x,p) < \varepsilon \}.$$

The <u>metric space topology</u> on X is

$$\mathcal{T} = \{ U \subseteq X \mid U \text{ is a union of open balls} \}.$$

(3) Let $(X, \mathcal{T})$ be a topological space.

Let $Y \subseteq X$. The <u>subspace topology</u> on Y is

$$\mathcal{T}_Y = \{ U \cap Y \mid U \in \mathcal{T} \}.$$

(4) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

The <u>product</u> of the sets X and Y is the set $X \times Y = \{ (x,y) \mid x \in X, y \in Y \}$.

The <u>product topology</u> on $X \times Y$ is

$$\mathcal{T}_{X \times Y} = \left\{ U \subseteq X \times Y \;\middle|\; \begin{array}{l} U \text{ is a union of } A \times B \\ \text{with } A \in \mathcal{T}_X \text{ and } B \in \mathcal{T}_Y \end{array} \right\}$$

Examples of open and closed sets.

Let $X = \mathbb{R}$ with the metric given by

$d(x,y) = |x-y|$ and the metric space topology.

Then

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\} = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$ is open

$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ is closed,

$[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$ is not open and not closed

(think of a door that is not open and not closed, i.e. ajar) and

$\emptyset$ and $\mathbb{R}$ are both open and closed.

Let X be a topological space and let $x \in X$.

A neighborhood of x is a subset $N \subseteq X$ such that there exists an open set U of X with

$x \in U$ and $U \subseteq N$.

The neighborhood filter of x is

$$\mathcal{N}(x) = \{\text{neighborhoods of } x\}$$

Let X be a topological space and let $E \subseteq X$.

The <u>interior</u> of E is the subset E° of X such that

(a) E° is open and $E^\circ \subseteq E$, and

(b) if U is open and $U \subseteq E$ then $U \subseteq E^\circ$.

The <u>closure</u> of E is the subset $\bar{E}$ of X such that

(a) $\bar{E}$ is closed and $\bar{E} \supseteq E$, and

(b) if V is closed and $V \supseteq E$ then $V \supseteq \bar{E}$.

In English:

E° is the largest open set contained in E and

$\bar{E}$ is the smallest closed set containing E.

An <u>interior point of E</u> is a point $x \in X$ such that there exists a neighborhood N of x such that $N \subseteq E$.

A <u>close point of E</u> is a point $x \in X$ such that if N is a neighborhood of x then $N \cap E \neq \emptyset$.

<u>Proposition</u> Let X be a topological space. Let $E \subseteq X$.

(a) The interior of E is the set of interior points of E.

(b) The closure of E is the set of close points of E.

<u>Proof of (a)</u>:

Let $I = \{x \in E \mid x \text{ is an interior point of } E\}$

To show: $I = E^\circ$.

To show: (aa) $I \subseteq E^\circ$

(ab) $E^\circ \subseteq I$.

(aa) Let $x \in I$.

Then there exists a neighborhood N of x with $N \subseteq E$.

So there exists an open set U with

$$x \in U \subseteq N \subseteq E.$$

Since $U \subseteq E$ and U is open $U \subseteq E^\circ$.

So $x \in E^\circ$.

So $I \subseteq E^\circ$.

(ab) To show: If $x \in E^\circ$ then $x \in I$.

Assume $x \in E^\circ$

Then E° is open and $E^\circ \subseteq E$.

So x is an interior point of E.

So $x \in I$

So $E^{o} \subseteq I$.

So $I = E^{o}$ //.