22.10.2014

Osmosis topics

Sets, Functions, Relations, Posets, $\mathbb{R}$.

Sets

elements, empty set, subset, union, intersection
disjoint, product of sets

Functions

injective, surjective, bijective,
equal functions, inverse function, restriction.
identity function, composition of functions.

Important Theorem

Let $f: S \to T$ be a function. An inverse function to f exists if and only if f is bijective.

Cardinality: isomorphism of sets.

finite, infinite, countable, uncountable.

HW Show that $\mathrm{Card}(\mathbb{Q}) = \mathrm{Card}(\mathbb{Z}) = \mathrm{Card}(\mathbb{Z}_{>0}) \neq \mathrm{Card}(\mathbb{R})$.

HW Show that $\mathrm{Card}(\mathbb{R}) = \mathrm{Card}(\mathbb{R}^2)$.

<u>Relations</u> Let S be a set.

A <u>relation on S</u> is a subset of $S \times S$.

Equivalence relation, partition of a set S.
Equivalence class.

<u>Important Theorem</u> Let S be a set.

(a) Let $\sim$ be an equivalence relation on S.
The set of equivalence classes of the relation $\sim$ is a partition of S.

(b) Let $\{S_\alpha\}$ be a partition of S.
The relation defined by

$s \sim t$ if s and t are in the same S_α

is an equivalence relation on S.

<u>Orders</u>

partially ordered set, totally ordered set, well ordered set.

upper/lower bound, $\sup(E)$, $\inf(E)$, $\min(E)$, $\max(E)$, maximal element, minimal element, smallest element, largest element.

Hasse diagram, lower/upper order ideal, intervals.

Ordered fields

An <u>ordered field</u> is a field $\mathbb{F}$ with a total order $\le$ such that

(a) If $a, b, c \in \mathbb{F}$ and $a \le b$ then $a+b \le b+c$,

(b) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $ab \ge 0$.

<u>Proposition</u> Let $(\mathbb{F}, \le)$ be an ordered field.

(a) If $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$.

(b) If $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$.

(c) If $a, b \in \mathbb{F}$ and $a > 0$ and $b > 0$ then $ab > 0$.

(d) If $a \in \mathbb{F}$ then $a^2 \ge 0$.

(e) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then

$$a \le b \text{ if and only if } a^2 \le b^2.$$

(f) $1 \ge 0$.

(g) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $a+b \ge 0$.

<u>HW</u> Show that $\mathbb{R}$ with the usual order is an ordered field.

<u>HW</u> Show that $\mathbb{C}$ is a field and there does not exist an order on $\mathbb{C}$ such that $\mathbb{C}$ is an ordered field.