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Convergence

§1 Limit points and cluster points.

§1.1 Filters, nets and sequences.

A *directed set* is a set P with a relation $\le$ such that

(a) If $i \in P$ then $i \le i$,

(b) If $i, j, k \in P$ and $i \le j$ and $j \le k$ then $i \le k$,

(c) If $i, j \in P$ then there exists $k \in P$ such that $i \le k$ and $j \le k$.

The favorite example of a directed set is

$\mathbb{Z}_{>0}$ with $i \le j$ if

there exists $n \in \mathbb{Z}_{\ge 0}$ with $i + n = j$.

(1.1) Definition Let X be a set.

- A *filter* on X is a collection $\mathcal{F}$ of subsets of X such that

  (a) (upper ideal) if $N \in \mathcal{F}$ and E is a subset of X with $N \subseteq E$ then $E \in \mathcal{F}$,

  (b) (closed under finite intersection) If $\ell \in \mathbb{Z}_{>0}$ and $N_1, N_2, \ldots, N_\ell \in \mathcal{F}$ then

  $$N_1 \cap N_2 \cap \cdots \cap N_\ell \in \mathcal{F},$$

  (c) $\emptyset \notin \mathcal{F}$.

- An <u>ultrafilter</u> is a maximal filter $\mathcal{F}$ on X (with respect to inclusion).

- A <u>net</u> on X is a function $\vec{x} : P \to X$, $n \mapsto x_n$, where P is a directed set.

- A <u>sequence</u> on X is a function $\vec{x} : \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$.

We often write $\vec{x} = (x_1, x_2, \ldots)$ for a sequence on X.

Let P be a directed set.

The <u>tail filter</u> is the filter on P given by

$$\{P_{\geq N} \mid N \in P\}, \quad \text{where } P_{\geq N} = \{j \in P \mid j \geq N\}.$$

Let Y be a topological space and let $y \in Y$.
The neighborhood filter of y is the filter on Y generated by the open sets containing y.

<u>(1.2) Definition</u> Let Y be a topological space and let $y \in Y$.

- A <u>neighborhood of y</u> is a subset $N \subseteq Y$ such that there exists an open set $U \subseteq Y$ with $y \in U \subseteq N$.

- The <u>neighborhood filter of y</u> is

$$\mathcal{N}(y) = \{\text{neighborhoods of } y\}.$$

(1.3) Definition Let Y be a topological space and let $\mathcal{G}$ be a filter on Y.

- A limit point of $\mathcal{G}$ is a point $y \in Y$ such that $\mathcal{G} \supseteq \mathcal{N}(y)$.
- A cluster point of $\mathcal{G}$ is a point $y \in Y$ such that if $N \in \mathcal{G}$ then $y \in \overline{N}$, where $\overline{N}$ is the closure of N.

(1.4) Definition Let Y be a topological space. Let X be a set and $f: X \to Y$ a function. Let $\mathcal{F}$ be a filter on X.

- A limit point of f with respect to $\mathcal{F}$ is a limit point of the filter on Y generated by $\{f(N) \mid N \in \mathcal{F}\}$.
- A cluster point of f with respect to $\mathcal{F}$ is a ~~limi~~ cluster point of the filter on Y generated by $\{f(N) \mid N \in \mathcal{F}\}$.

Write

$$y = \lim_{\mathcal{F}} f \quad \text{if } y \text{ is a limit point}$$

of f with respect to $\mathcal{F}$.

Let X and Y be topological spaces and let $f: X \to Y$ be a function. Let $a \in X$. Write

$$y = \lim_{x \to a} f(x) \quad \text{if } y \text{ is a limit point}$$

of f with respect to the neighborhood filter of a.

Let $\vec{x}: \begin{matrix} P & \to & X \\ n & \mapsto & x_n \end{matrix}$ be a net on X. Write

$$y = \lim_{n \to \infty} x_n \quad \text{if } y \text{ is a limit point}$$

of $\vec{x}$ with respect to the tail filter on P.

Let $\vec{x} = (x_1, x_2, \ldots)$ be a sequence in X. Write

$$y = \lim_{n \to \infty} x_n, \quad \text{if } y \text{ is a limit point}$$

of $\vec{x}: \begin{matrix} \mathbb{Z}_{>0} & \to & X \\ n & \mapsto & x_n \end{matrix}$ with respect to the tail filter on $\mathbb{Z}_{>0}$.

A <u>cluster point</u> of a sequence $\vec{x} = (x_1, x_2, \ldots)$ in X is a cluster point of $\vec{x}: \begin{matrix} \mathbb{Z}_{>0} & \to & X \\ n & \mapsto & x_n \end{matrix}$ with respect to the tail filter on $\mathbb{Z}_{>0}$.

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<u>Proposition</u> Let X be a topological space.

(a) Let $A \subseteq X$. Then

$$\bar{A} = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a filter } \mathcal{F} \text{ on } X \text{ with} \\ A \in \mathcal{F} \text{ and } \lim\limits_{\mathcal{F}} = z \end{array} \right\}$$

$$= \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a net } \vec{a}: P \to A \\ \text{with } \lim\limits_{n \to \infty} a_n = z \end{array} \right\}$$

(b) Let Y be a topological space and let

$$f: X \to Y \text{ be a function.}$$

The following are equivalent.

(1) $f: X \to Y$ is continuous.

(2) If $\mathcal{F}$ is a filter on X and $\lim\limits_{\mathcal{F}}$ exists then

$$f\left(\lim_{\mathcal{F}}\right) = \lim_{\mathcal{F}} f.$$

(3) If $a \in X$ then $\lim\limits_{x \to a} f(x) = f(a)$

(4) If $\begin{array}{l}\vec{x}: P \to X \\ n \mapsto x_n\end{array}$ is a net on X and $\lim\limits_{n \to \infty} x_n$ exists then

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right).$$

Notes and References: See [Clark, Cor. 5.9 and Prop. 5.14].

A topological space X is <u>first countable</u> if X satisfies:

if $a \in X$ then there exists a countable collection of neighborhoods of x which generates $\mathcal{N}(a)$.

<u>Proposition</u> Let X be a first countable topological space.

(a) ~~If~~ Let $A \subseteq X$. Then

$$\bar{A} = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a sequence } (a_1, a_2, \ldots) \\ \text{in } A \text{ with } \lim_{n\to\infty} a_n = z \end{array} \right\}$$

(b) Let Y be a topological space and let $f: X \to Y$ be a function.

The following are equivalent

(1) f is continuous

(2) If $\vec{x} = (x_1, x_2, \ldots)$ is a sequence ~~and~~ in X and $\lim_{n\to\infty} x_n$ exists then

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

<u>Proposition</u> Let X be a topological space and let $(x_1, x_2, \ldots)$ be a sequence in X.

(a) If $(x_{n_1}, x_{n_2}, \ldots)$ is a subsequence of $(x_1, x_2, \ldots)$ and $y = \lim_{k \to \infty} x_{n_k}$ exists then y is a cluster point of $(x_1, x_2, \ldots)$.

(b) If X is first countable and y is a cluster point of $(x_1, x_2, \ldots)$ then there exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ of $(x_1, x_2, \ldots)$ such that $y = \lim_{k \to \infty} x_{n_k}$.

Example Let X be an uncountable set and let

$$\mathcal{T} = \{ A \subseteq X \mid A^c \text{ is countable} \}.$$

Show that

(a) X is a topological space.

(b) X is not first countable.

(c) X is not Hausdorff.

(d) X is not discrete

(e) If $(x_1, x_2, \ldots)$ is a sequence in X and $y = \lim_{n\to\infty} x_n$ exists then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $x_n = y$ i.e., $(x_1, x_2, \ldots)$ is eventually constant at y.

(f) If $A \subseteq X$ and A is uncountable then $\overline{A} = X$.

(g) If $A \subseteq X$ then

$$\left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a sequence} \\ (a_1, a_2, \ldots) \text{ in } A \text{ with } z = \lim_{n\to\infty} a_n \end{array} \right\} = A$$

(h) If $A \subseteq X$ is uncountable and $A \neq X$ then

$$\overline{A} \neq \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a sequence} \\ (a_1, a_2, \ldots) \text{ in } A \text{ with } z = \lim_{n\to\infty} a_n \end{array} \right\}.$$

Notes and References: [Clark Example 2.1.5]