Examples of spaces

Subspaces and product spaces, $B(V,W)$

Function spaces.

Subspaces

HW (a) Let X be a normed vector space.

Let $V \subseteq X$ be a subspace. Show that V is a normed vector space with the same norm [Bressan, Ch2P4].

(b) Let (X,d) be a metric space. Let $Y \subseteq X$ be a subset. Show that (Y,d) is a metric space [Rubinstein Example 2 to Def. 2.1].

(c) State and prove similar statements for topological spaces and for inner product spaces (see Rubinstein Theorem 2.21

Product spaces

HW Let X and Y be Banach spaces.

(a) Prove that $X \times Y$ with

$$\|(x,y)\| = \max\{\|x\|, \|y\|\}$$

is a Banach space [Bressan Ch2P2]

(b) Prove that $X \times Y$ with

$$\|(x,y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}$$

is a normed vector space. Is it a Banach space?

(c) Are $X \times Y$ from (a) and $X \times Y$ from (b) the same as topological spaces? [Bressan Ch5 Ex 2].

__HW__ Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $Y = X_1 \times X_2$ and define

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

$$\rho((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

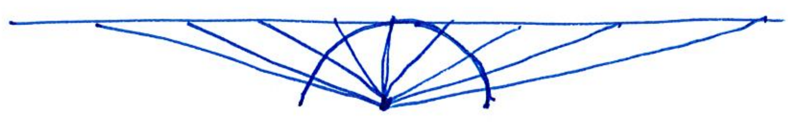
$$\sigma((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$$

(a) Show that (Y, d), (Y, ρ) and (Y, σ) are metric spaces.

(b) Show that (Y, d), (Y, ρ) and (Y, σ) are the same as topological spaces.

[Rubinstein Example (3) to Def. 2.1 and Example 2 to Defn. 2.11].

__Example__ $(0,1)$ is homeomorphic to $\mathbb{R}$.



$(0,1)$ is bounded

$\mathbb{R}$ is not bounded

$(0,1)$ is not complete

$\mathbb{R}$ is complete.

So boundedness and completeness are not topological properties.

Let X be a set and let $d_1: X \times X \to \mathbb{R}_{\geq 0}$ and $d_2: X \times X \to \mathbb{R}_{\geq 0}$ be metrics on X.

The metrics d_1 and d_2 are <u>Lipschitz equivalent</u> if there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that

if $x, y \in X$ then $c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y)$.

<u>Proposition</u> Let X be a set and let d_1 and d_2 be Lipschitz equivalent metrics on X. Show that d_1 and d_2 produce the same topology on X.

<u>Sketch of proof</u>

To show: If $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then

$B^2_\varepsilon(x) = \{y \in X \mid d_2(x,y) < \varepsilon\}$ is d_1-open.

Assume $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that $B^1_\delta(x) \subseteq B^2_\varepsilon(x)$

where $B^1_\delta(x) = \{y \in X \mid d_1(x,y) < \delta\}$.

Let $\delta = c_1 \varepsilon$.