Metric and Hilbertspaces 14.10.2014

Example Consider the sequence

$e_1, e_2, e_3, \ldots$ in ℓ^p where

$e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 in the ith spot.

Then

(a) $e_1, e_2, e_3, \ldots$ has no convergent subsequence.
(the closed unit ball is not compact).

(b) e_1, e_2, e_3 weakly converges to 0

i.e. If $x \in \ell^2$, $x = (x_1, x_2, \ldots)$, then

$\langle x, e_1\rangle, \langle x, e_2\rangle, \langle x, e_3\rangle$ converges to 0.

(c) Let $I: \ell^p \to \ell^p$ be the identity operator.

Then $Ie_1, Ie_2, Ie_3, \ldots$

is the sequence $e_1, e_2, e_3, \ldots$ has no convergent subsequence and so $\overline{I(S)}$ is not compact.

Let $T: H \to H$ be a compact linear operator.

Let λ be an nonzero eigenvalue of T and let

$X_\lambda = \{x \in H \mid Tx = \lambda x\}$.

If $\dim(X_\lambda)$ is infinite then there is a sequence

$e_1, e_2, e_3, \ldots$ in X_λ

of linearly independent vectors with

$\|e_j\| = 1$ and $Te_j = \lambda e_j$ and $\langle e_i, e_j\rangle = \delta_{ij}$.

Then $\|Te_m - Te_n\|^2 = \|\lambda e_m - \lambda e_n\|^2 = |\lambda|^2 \|e_m - e_n\|^2$

$$= |\lambda|^2 (\|e_m\|^2 + \|e_n\|^2)$$
$$= |\lambda|^2 \cdot 2$$

So the sequence $Te_1, Te_2, \ldots$ has no Cauchy subsequence and no convergent subsequence.

So T is not compact //.

Let $T: H \to H$ be a compact linear operator and let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be distinct eigenvalues of T.

Assume $\lim_{k\to\infty} \lambda_k \neq 0$.

Then there is a subsequence $\lambda_{k_1}, \lambda_{k_2}, \ldots$ and $C \in \mathbb{R}_{>0}$ with $|\lambda_{k_j}| > C$ for $j \in \mathbb{Z}_{>0}$.

Let $e_1, e_2, \ldots$ be such that $\|e_i\| = 1$ and $Te_j = \lambda_{k_j} e_j$. Since $\lambda_{k_1}, \lambda_{k_2}, \ldots$ are all distinct then $\langle e_i, e_j \rangle = 0$.

So
$$\|Te_m - Te_n\|^2 = \|\lambda_m e_m - \lambda_n e_n\|^2 = \langle \lambda_m e_m - \lambda_n e_n, \lambda_m e_m - \lambda_n e_n \rangle$$
$$= \boxed{\cancel{\lambda_m}}\ \lambda_m^2 \|e_m\|^2 + \lambda_n^2 \|e_n\|^2$$
$$> 2C^2$$

So $Te_1, Te_2, Te_3, \ldots$ has no Cauchy subsequence and no convergent subsequence.

So T is not compact. //

<u>Remark</u> Let H be a Hilbert space and let $T: H \to H$ be a compact operator. If H is infinite dimensional then T is not a bijection (or, better, $0 \cdot I - T$ is not a bijection).

Let $T: H \to H$ be a compact self-adjoint operator. For each eigenvalue let B_λ be an orthonormal basis of X_λ.

and let

$$B = \bigcup_{\text{eigenvalues}} B_\lambda$$

and let

$$X = \overline{\text{span}(B)}.$$

Then

$$H = X \oplus X^\perp \quad \text{and} \quad T = T_X \oplus T_{X^\perp}$$

with T_X and $T_{X^\perp}$ both compact operators.

$$T_{X^\perp}: X^\perp \to X^\perp.$$

Then $T_{X^\perp}$ has an eigenvector with eigenvalue $\|T_{X^\perp}\|$.
But all eigenspaces of T are in X.

So $H = X$.