Self adjoint operators

Let H be an inner product space.

Let

$T: H \to H$ be a self adjoint operator.

1(a) If $x \in H$ is an eigenvector for T then the eigenvalue of x is in $\mathbb{R}$.

Assume $x \in H$ and $Tx = \lambda x$. Then

$$\begin{aligned}\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle &= \langle x, T^* x\rangle \\ &= \langle x, Tx\rangle, \text{ since } T \text{ is selfadjoint,} \\ &= \langle x, \lambda x\rangle = \bar{\lambda}\langle x, x\rangle.\end{aligned}$$

If $x \neq 0$ then $\langle x, x\rangle \neq 0$ and $\lambda = \bar{\lambda}$.

So $\lambda \in \mathbb{R}$.

(b) Assume $\lambda \neq \gamma$. Let λ and γ be eigenvalues of T and let

$$X_\lambda = \{x \in H \mid Tx = \lambda x\} \text{ and } X_\gamma = \{x \in H \mid Tx = \gamma x\}.$$

Then X_λ is orthogonal to X_γ.

Proof: To show: If $x \in X_\lambda$ and $y \in X_\gamma$ then $\langle x, y\rangle = 0$.

Assume $\lambda \neq \gamma$.

Assume $x \in X_\lambda$ and $y \in X_\gamma$. Then

$$\begin{aligned}\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle &= \langle x, T^* y\rangle \\ &= \langle x, Ty\rangle \text{ since } T \text{ is self adjoint} \\ &= \langle x, \gamma y\rangle = \bar{\gamma}\langle x, y\rangle = \gamma\langle x, y\rangle, \text{ since } \gamma \in \mathbb{R}.\end{aligned}$$

So $(\lambda-\gamma)\langle x,y\rangle = 0$.

So $\lambda-\gamma=0$ or $\langle x,y\rangle=0$.

Since $\lambda\neq\gamma$ then $\langle x,y\rangle=0$.

(c) Let H be a Hilbert space and let $T: H\to H$ be a bounded selfadjoint operator. Let

$$m = \inf\{\langle Tu,u\rangle \mid \|u\|=1\} \quad\text{and}$$

$$M = \sup\{\langle Tu,u\rangle \mid \|u\|=1\}.$$

Then $\|T\| = \max\{-m, M\}$.

<u>Proof</u> Assume $|m|\le M$ (otherwise replace Λ by $-\Lambda$).

If $u,v\in H$ then

$$\begin{aligned}4\langle Tu,v\rangle &= \langle T(u+v),u+v\rangle - \langle T(u-v),u-v\rangle \\ &\le M\left(\|u+v\|^2+\|u-v\|^2\right) \\ &= 2M(\|u\|^2+\|v\|^2).\end{aligned}$$

If $Tu\neq 0$ set

$$v = \frac{Tu}{\|Tu\|}\cdot\|u\|.$$

Then

$$2\|u\|\,\|Tu\| = 2\langle Tu,v\rangle \le M(\|u\|^2+\|v\|^2) = 2M\|u\|^2.$$

So

$$\|Tu\|\le M\|u\|, \text{ for all } u\in H.$$

So $\|T\|\le M$.

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Assume $u \in H$ and $\|u\|=1$.

To show: $\|T\| \geq |\langle Tu, u\rangle|$.

$$|\langle Tu, u\rangle| \leq \|Tu\| \cdot \|u\|, \text{ by Cauchy-Schwarz}$$
$$\leq \|T\|,$$

since $\|T\| = \sup\left\{ \frac{\|Tu\|}{\|u\|} \;\middle|\; u \in H \right\}$

$$= \sup\{ \|Tu\| \mid \|u\| = 1\}.$$

So $\|T\| \geq M$.

So $\|T\| \leq M$.

(d) Let $T: H \to H$ be a compact linear operator.
Assume $\lambda \neq 0$.
Let $X_\lambda = \{x \in H \mid Tx = \lambda x\}$.

Then $\dim(X_\lambda)$ is finite.

~~Proof~~ Proof by contradiction.

Assume $\dim(X_\lambda)$ is infinite dimensional.

Let $e_1, e_2, \ldots$ be an orthonormal sequence in X_λ. Then

$$\|e_m - e_n\|^2 = \|e_m\|^2 + \|e_n\|^2 = 2$$

and $\|Te_m - Te_n\| = \|\lambda e_m - \lambda e_n\|^2 = |\lambda|^2 \cdot 2 = 2|\lambda|^2$

So $Te_1, Te_2, \ldots$ does not have a convergent subsequence.

<u>Theorem</u> Let T be a nonzero self adjoint compact operator $T: H \to H$. Then there exists an orthonormal basis of eigenvectors of T.

<u>Proof</u> Let $X_\lambda = \{x \in H \mid Tx = \lambda x\}$.

(A) If $\lambda \neq \gamma$ ~~and $\lambda \neq 0$ and $\gamma \neq 0$~~ then $X_\lambda \perp X_\gamma$.

<u>Proof</u> Let $x \in X_\lambda$ and $y \in X_\gamma$. Then

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, T^* y\rangle$$

$= \langle x, Ty\rangle$ since T is self adjoint

$= \langle x, \gamma y\rangle = \bar{\gamma}\langle x, y\rangle$

$= \gamma\langle x, y\rangle$ since eigenvalues of a self adjoint operator ~~are~~ in $\mathbb{R}$ ~~real~~.

So $(\lambda - \gamma)\langle x, y\rangle = 0$.

So $\lambda - \gamma = 0$ or $\langle x, y\rangle = 0$.

So $\lambda = \gamma$ or $\langle x, y\rangle = 0$.

(A') If $\lambda \neq 0$ then $\dim X_\lambda < \infty$

(B) Choose an orthonormal basis B_λ of X_λ.

Let

$$B = \bigcup_{\lambda \in \Lambda} B_\lambda \quad \text{where } \Lambda = \{\text{eigenvalues of } T\}$$

To show: $H = \overline{\operatorname{span} B}$.