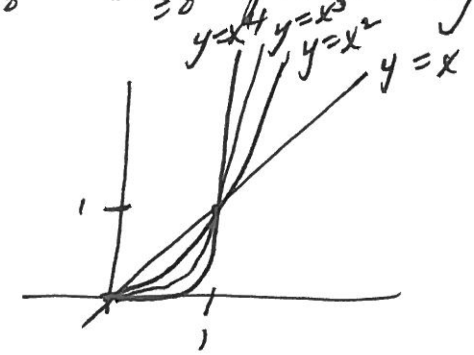
Hölder, Minkowski, Cauchy-Schwarz and triangle inequalities

Let $q \in \mathbb{R}_{\ge 1}$. Let $p \in \mathbb{R}_{>1} \cup \{\infty\}$ be given by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The functions $h: \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ given by

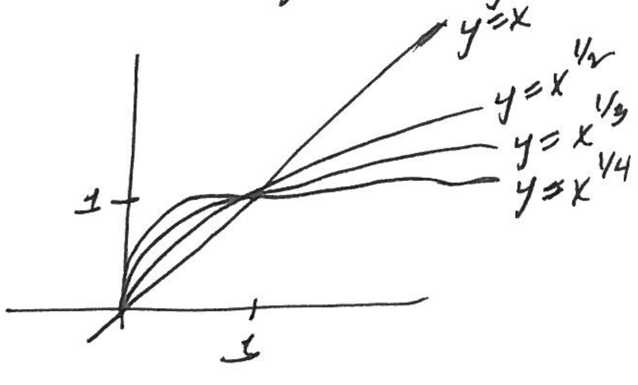
$$h(x) = x^q$$



are increasing with $h(1) = 1$.

The functions $h: \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ given by

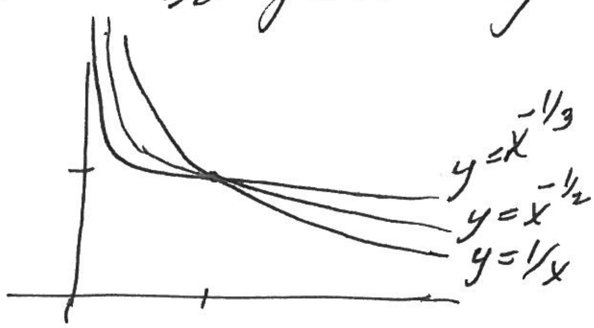
$$h(x) = x^{1/q}$$



are increasing with $h(1) = 1$.

The functions $h: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ given by

$$h(x) = x^{-1/q}$$



are decreasing with $h(1) = 1$.

Thus the functions $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by

$g(x) = x^{-1/q} - 1$ are decreasing with $g(1) = 0$.

So
$$x^{-1/q} - 1 \leq 0 \quad \text{for } x \in \mathbb{R}_{\geq 1}.$$

If $f : \mathbb{R}_{>0} \to \mathbb{R}$ is given by

$$f(x) = x^{1/p} - \frac{1}{p} x$$

then $\frac{df}{dx} = \frac{1}{p} x^{\frac{1}{p}-1} - \frac{1}{p} = \frac{1}{p}\left(x^{-1/q} - 1\right)$ and so

f is decreasing for $x \in \mathbb{R}_{>1}$ and $f(1) = 1 - \frac{1}{p} = \frac{1}{q}$

So
$$x^{1/p} - \frac{1}{p} x \leq \frac{1}{q} \quad \text{for } x \in \mathbb{R}_{\geq 1}$$

Let $a, b \in \mathbb{R}_{>0}$ with $a \geq b$ and let $x = \frac{a}{b}$.

Then
$$\frac{1}{q} \geq \left(\frac{a}{b}\right)^{\frac{1}{p}} - \frac{1}{p}\left(\frac{a}{b}\right) = \frac{1}{b}\left(a^{\frac{1}{p}} b^{-\frac{1}{p}+1} - \frac{1}{p} a\right) = \frac{1}{b}\left(a^{\frac{1}{p}} b^{1/q} - \frac{1}{p} a\right).$$

So
$$\frac{1}{p} a + \frac{1}{q} b \geq a^{\frac{1}{p}} b^{\frac{1}{q}} \quad \text{for } a, b \in \mathbb{R}_{>0}$$

Let $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $y=(y_1,\dots,y_n)\in\mathbb{R}^n$.
Then

$$\frac{|x_i y_i|}{\|x\|_p\,\|y\|_q} \le \frac{1}{p}\left(\frac{|x_i|}{\|x\|_p}\right)^p + \frac{1}{q}\left(\frac{|y_i|}{\|y\|_q}\right)^q$$

So

$$\sum_{i=1}^n \frac{|x_i y_i|}{\|x\|_p\|y\|_q} \le \sum_{i=1}^n\left(\frac{1}{p}\left(\frac{|x_i|}{\|x\|_p}\right)^p + \frac{1}{q}\left(\frac{|y_i|}{\|y\|_q}\right)^q\right)$$

$$= \frac{1}{p}+\frac{1}{q}=1$$

So

$$\sum_{i=1}^n |x_i y_i| \le \|x\|_p\,\|y\|_q$$

So $$\left|\sum_{i=1}^n x_i y_i\right| \le \sum_{i=1}^n |x_i y_i| \le \|x\|_p\,\|y\|_q .$$

Using

$$|x_i+y_i| \le |x_i| + |y_i| \quad \text{and} \quad p-1 = p\left(1-\frac{1}{p}\right) = p\,\frac{1}{q} = \frac{p}{q}$$

and

$$\left\| |x_1+y_1|^{p/q}, \dots, |x_n+y_n|^{p/q} \right\|_q = \left(\sum_{i=1}^{n} \left(|x_i+y_i|^{p/q}\right)^q \right)^{1/q}$$

$$= \left(\sum_{i=1}^{n} |x_i+y_i|^p \right)^{\frac{1}{p}\cdot\frac{p}{q}} = \left(\|x+y\|_p\right)^{p/q},$$

gives

$$\|x+y\|_p^p = \sum_{i=1}^{n} |x_i+y_i|^p = \sum_{i=1}^{n} |x_i+y_i|\,|x_i+y_i|^{p-1}$$

$$\le \sum_{i=1}^{n} \left(|x_i|+|y_i|\right) |x_i+y_i|^{p/q}$$

$$= \sum_{i=1}^{n} |x_i|\,|x_i+y_i|^{p/q} + \sum_{i=1}^{n} |y_i|\,|x_i+y_i|^{p/q}$$

$$\le \|x\|_p \left\|\left(|x_1+y_1|^{p/q}, \dots, |x_n+y_n|^{p/q}\right)\right\|_q + \|y\|_p \left\|\left(|x_1+y_1|^{p/q}, \dots, |x_n+y_n|^{p/q}\right)\right\|_q$$

$$= \|x\|_p \,\|x+y\|_p^{p/q} + \|y\|_p \,\|x+y\|_p^{p/q}$$

$$= \left(\|x\|_p + \|y\|_p\right) \|x+y\|_p^{p-1}.$$

~~Then~~ Dividing both sides by $\|x+y\|_p^{p-1}$, then

$$\|x+y\|_p \le \|x\|_p + \|y\|_p.$$

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $p \in \mathbb{R}_{\geq 1}$ define

$$\|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}.$$

~~Let~~ For $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ define

$$|x| = \|x\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2} = \sqrt{x_1^2 + \cdots + x_n^2}$$

and, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ define

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n.$$

<u>Theorem</u> Let $x, y \in \mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. If $p \in \mathbb{R}_{>1}$ and q is given by $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \|x\|_p \|y\|_q \quad \text{and} \quad \|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

<u>Corollary</u> Let $x, y \in \mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$|\langle x, y \rangle| = \left|\sum_{i=1}^{n} x_i y_i\right| \leq |x||y| \quad \text{and} \quad |x+y| \leq |x| + |y|.$$

<u>Special case</u>: Let $x, y \in \mathbb{R}$. Then

$$|xy| = |x||y| \quad \text{and} \quad |x+y| \leq |x| + |y|.$$