<u>Theorem 13.3</u> Let H be a Hilbert space.

Let $T: H \to H$ be a bounded self adjoint

Then
$$\|T\| = \sup\{|\langle Tx, x\rangle| \mid \|x\|=1\}.$$

<u>Proof</u> To show: (a) $\|T\| \geq \sup\{|\langle Tx,x\rangle| \mid \|x\|=1\}$.

(b) $\|T\| \leq \sup\{|\langle Tx,x\rangle| \mid \|x\|=1\}$.

(a) Assume $x \in H$ and $\|x\|=1$.

To show: $\|T\| \geq |\langle Tx, x\rangle|$.

$$\begin{aligned} |\langle Tx,x\rangle| &\leq \|Tx\|\,\|x\|, \quad \text{by Cauchy-Schwarz} \\ &\leq \|T\|, \quad \text{by definition } \|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} \,\middle|\, x\in H\right\} \\ &\qquad\qquad = \sup\{\|Tx\| \mid \|x\|=1\}. \end{aligned}$$

(b) Let $x \in H$ with $Tx \neq 0$ and $\|x\|=1$.

Let $y = \dfrac{Tx}{\|Tx\|}$ and let $\beta = \sup\{|\langle Tx,x\rangle| \mid \|x\|=1\}$.

Then

$$\|Tx\| = \frac{\langle Tx, Tx\rangle}{\|Tx\|} = \langle Tx, y\rangle = \mathrm{Re}\langle Tx, y\rangle$$

$$= \frac{1}{4}\left(4\,\mathrm{Re}\langle Tx, y\rangle\right) = \frac{1}{4}\left(2\langle Tx, y\rangle + 2\,\overline{\langle Tx, y\rangle}\right)$$

$$= \frac{1}{4}\left(2\langle Tx, y\rangle + 2\langle y, Tx\rangle\right)$$

$$= \frac{1}{4}\left(2\langle Tx, y\rangle + 2\langle y, T^{*}x\rangle\right)$$

$$= \frac{1}{4}\left(2\langle Tx, y\rangle + 2\langle Ty, x\rangle\right), \quad \text{since } T \text{ is self adjoint,}$$

$$= \frac{1}{4}\left(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right)$$

$$\leq \frac{1}{4}\left|\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right|$$

$$\leq \frac{1}{4}\left(\left|\langle T(x+y), x+y\rangle\right| + \left|\langle T(x-y), x-y\rangle\right|\right)$$

$$\leq \frac{1}{4}\left(\left|\left\langle \frac{T(x+y)}{\|x+y\|}, \frac{x+y}{\|x+y\|}\right\rangle\right| \|x+y\|^2 + \left|\left\langle \frac{T(x-y)}{\|x-y\|}, \frac{x-y}{\|x-y\|}\right\rangle\right| \|x-y\|^2\right)$$

$$\leq \frac{1}{4}\left(\beta\|x+y\|^2 + \beta\|x-y\|^2\right)$$

$$= \frac{1}{4}\beta\left(\langle x+y, x+y\rangle + \langle x-y, x-y\rangle\right)$$

$$= \frac{1}{4}\beta\left(\|x\|^2 \cancel{+ 2\langle x,y\rangle + 2\langle x,y\rangle} + 2\,\mathrm{Re}\langle x,y\rangle - 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2\right)$$
$$+ \|x\|^2 + \|y\|^2$$

$$= \frac{1}{4}\beta\left(\|x\|^2 + \|y\|^2\right)\cdot 2$$

$$= \frac{1}{4}\beta(1+1)\cdot 2 = \frac{1}{4}\beta\cdot 2\cdot 2 = \beta = \sup\{|\langle Tx, x\rangle| \mid \|x\| = 1\}.$$

<u>Theorem</u> Let H be a Hilbert space and let $T: H \to H$ be a nonzero self adjoint compact operator. Then

there exists $x \in H$ such that $\|x\| = 1$ and

if $u \in H$ and $\|u\| = 1$ then $\|\langle Tu, u\rangle\| \le |\langle Tx, x\rangle|$.

<u>Proof</u> By Theorem 13.3

$$\|T\| = \sup\{ |\langle Tu, u\rangle| \mid \|u\| = 1\}.$$

Let $x_1, x_2, \ldots \in H$ with $\|x_n\| = 1$ and

$$\lim_{n\to\infty} |\langle Tx_n, x_n\rangle| = \|T\|.$$

Then $x_{n_1}, x_{n_2}, \ldots$ be a subsequence of $x_1, x_2, \ldots$ such that

$$\lim_{k\to\infty} \langle Tx_{n_k}, x_{n_k}\rangle \quad \text{exists.}$$

Use that T is compact to find a subsequence $x_{n_{k_1}}, x_{n_{k_2}}, \ldots$ of $x_{n_1}, x_{n_2}, \ldots$ such that

$w = \lim_{j\to\infty} Tx_{n_{k_j}}$ exists. Let $x = \dfrac{w}{\|w\|}$

To show: (a) $|\langle Tx, x\rangle| = \|T\|$

(b) $Tx = \lambda x$ with $\|T\| = \lambda$.

Let $\lambda = \|T\|$.

Since

$$0 \le \|(\lambda I - T)(x_{n_{k_j}})\|^2$$
$$= \|\lambda x_{n_{k_j}} - T x_{n_{k_j}}\|^2$$
$$= \langle \lambda x_{n_{k_j}} - T x_{n_{k_j}}, \lambda x_{n_{k_j}} - T x_{n_{k_j}} \rangle$$
$$= \lambda^2 \|x_{n_{k_j}}\| - \lambda \langle x_{n_{k_j}}, T x_{n_{k_j}} \rangle - \langle T x_{n_{k_j}}, \lambda x_{n_{k_j}} \rangle + \|T x_{n_{k_j}}\|^2$$
$$= \lambda^2 \|x_{n_{k_j}}\| - 2\lambda \langle x_{n_{k_j}}, T x_{n_{k_j}} \rangle + \|T x_{n_{k_j}}\|^2$$
$$\le \lambda^2 - 2\lambda \langle x_{n_{k_j}}, T x_{n_{k_j}} \rangle + \|T\|^2.$$

Since the right hand side approaches

$$\lambda^2 - 2\lambda^2 + \|T\|^2 = 0 \quad \text{as } j \to \infty$$

then

$$\lim_{j \to \infty} \|(\lambda I - T)(x_{n_{k_j}})\|^2 = 0 \text{ so that } \lim_{j \to \infty} (\lambda I - T)(x_{n_{k_j}}) = 0.$$

So
$$\lambda w = \lim_{j \to \infty} \lambda T x_{n_{k_j}} = \lim_{j \to \infty} T(\lambda x_{n_{k_j}})$$
$$= \lim_{j \to \infty} T((\lambda I - T) x_{n_{k_j}} + T x_{n_{k_j}})$$
$$= T(0 + w) = Tw.$$