Bessel's inequality Metric and Hilbert Spaces (1B)

Let H be a Hilbert space. 26.09.2014

Let $a_1, a_2, a_3, \ldots$ be an orthonormal sequence in H.

To show: $\sum_{n\in\mathbb{Z}_{>0}} |\langle x, a_n\rangle|^2 \le \|x\|^2$.

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To show: $\lim_{k\to\infty}\left(\sum_{n=1}^{k}\langle x, a_n\rangle^2\right) \le \|x\|^2$.

To show: If $k\in\mathbb{Z}_{>0}$ then $\sum_{n=1}^{k}\langle x, a_n\rangle^2 \le \|x\|^2$.

Assume $k\in\mathbb{Z}_{>0}$. Let

$$x_k = \sum_{n=1}^{k}\langle x, a_n\rangle a_n \quad \text{so that } \|x_k\|^2 = \sum_{n=1}^{k}\langle x, a_n\rangle^2.$$

To show: $\|x_k\|^2 \le \|x\|^2$.

Then

$$\begin{aligned}\langle x - x_k, x_k\rangle &= \langle x, x_k\rangle - \langle x_k, x_k\rangle \\ &= \sum_{n=1}^{k}\langle x, a_n\rangle\langle x, a_n\rangle - \sum_{n=1}^{k}\langle x_k, x_k\rangle \\ &= 0, \quad \text{and}\end{aligned}$$

$$\begin{aligned}\|x\|^2 = \langle x, x\rangle &= \langle x_k + (x - x_k), x_k + (x - x_k)\rangle \\ &= \langle x_k, x_k\rangle + \langle x_k, (x - x_k)\rangle + \langle (x - x_k), x_k\rangle + \\ &\quad + \langle x - x_k, x - x_k\rangle \\ &= \|x_k\|^2 + 0 + 0 + \|x - x_k\|^2.\end{aligned}$$

So $\|x_k\|^2 \le \|x\|^2$.

Let $W = \text{span}\{a_1, a_2, \ldots\}$.

<u>To show:</u> $P: H \to H$ given by

$$P(x) = \sum_{n \in \mathbb{Z}_{>0}} \langle x, a_n \rangle a_n$$

is an orthogonal projection onto $\overline{W}$.

To show: (a) $P: H \to H$ is a function

(b) If $x \in H$ then $P(x) \in \overline{W}$

(c) If $x \in H$ then $x - P(x) \in (\overline{W})^\perp$.

(a) To show: If $x \in H$ then $P(x) = \sum_{n \in \mathbb{Z}_{>0}} \langle x, a_n \rangle a_n$ exists in H.

Assume $x \in H$.

Let

$$x_k = \sum_{n=1}^{k} \langle x, a_n \rangle a_n.$$

To show: $\lim_{k \to \infty} x_k$ exists in H.

Since H is complete, we need

To show: $x_1, x_2, x_3, \ldots$ is a Cauchy sequence in H.

We know: $\|x_k\| = \sum_{n=1}^{k} \langle x, a_n \rangle^2$ so that

$\|x_1\|, \|x_2\|, \ldots$ is an increasing sequence in $\mathbb{R}_{\geq 0}$ bounded by $\|x\|$, (by Bessel's inequality).

So $\|x_1\|, \|x_2\|, \ldots$ converges. Let $y = \lim_{k \to \infty} \|x_k\|$

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq N}$ then $\|x_m - x_n\| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq N}$ then $\|x_m - x_n\| < \varepsilon$.

Let $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq 0}$ then $|y^2 - \|x_k\|^2| < \varepsilon/2$

To show: If $m, n \in \mathbb{Z}_{\geq N}$ then $\|x_m - x_n\| < \varepsilon$.

Assume $m, n \in \mathbb{Z}_{\geq N}$

To show: $\|x_m - x_n\| < \varepsilon$.

$$\|x_m - x_n\|^2 = \left\| \sum_{j=1}^{m} \langle x, a_j \rangle a_j - \sum_{j=1}^{n} \langle x, a_j \rangle a_j \right\|^2$$

$$= \left\| \sum_{j=m+1}^{n} \langle x, a_j \rangle a_j \right\|^2 = \sum_{j=m+1}^{n} \langle x, a_j \rangle^2$$

$$= \left| \|x_n\|^2 - \|x_m\|^2 \right|$$

$$= \left| \|x_n\|^2 - y^2 + y^2 - \|x_m\|^2 \right|$$

$$\leq \left| \|x_n\|^2 - y^2 \right| + \left| y^2 - \|x_m\|^2 \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $x_1, x_2, \ldots$ is a Cauchy sequence in H.

So $\lim_{n \to \infty} x_n$ exists in H.

So $\sum_{j \in \mathbb{Z}_{>0}} \langle x, a_j \rangle a_j$ exists in H.

(b) To show: If $x \in H$ then $P(x) \in \overline{W}$.

Assume $x \in X$.

To show: $\sum_{n \in \mathbb{Z}_{>0}} \langle x, a_n \rangle a_n \in \overline{W}$.

To show: $\lim_{k \to \infty} x_k \in \overline{W}$.

To show: $x_k \in W$.

Since $x_k = \sum_{j=1}^{k} \langle x, a_j \rangle a_j \in \mathrm{span}\{a_1, a_2, \ldots\}$

then $x_k \in W$.

So $P(x) = \lim_{k \to \infty} x_k \in \overline{W}$.

(c) To show: If $x \in H$ then $x - P(x) \in \overline{W}^{\perp}$.

Assume $x \in H$

To show: $x - P(x) \in \overline{W}^{\perp}$

To show: If $b \in \overline{W}$ then $\langle x - P(x), b \rangle = 0$.

Assume $b \in \overline{W}$.

Let $b_1, b_2, \ldots$ be a sequence in W with $\lim_{n\to\infty} b_n = b$.

To show: $\langle x - P(x), b \rangle = 0$.

$$\langle x - P(x), b \rangle = \langle x - P(x), \lim_{n\to\infty} b_n \rangle$$

$$= \lim_{n\to\infty} \langle x - P(x), b_n \rangle \quad \left(\begin{array}{l}\text{since} \\ \langle x - P(x), \cdot \rangle : H \to \mathbb{C} \\ \text{is continuous}\end{array}\right)$$

and there exist $\ell \in \mathbb{Z}_{>0}$ and $c_1, \ldots, c_\ell \in \mathbb{C}$ such that

$$\langle x - P(x), b_n \rangle = \sum_{k=1}^{\ell} c_k \langle x - P(x), a_k \rangle.$$

Then

$$\langle x - P(x), a_j \rangle = \langle x - \lim_{k\to\infty} x_k, a_j \rangle$$

$$= \lim_{k\to\infty} \langle x - x_k, a_j \rangle \quad \left(\begin{array}{l}\text{since} \\ \langle \cdot, a_j \rangle : H \to \mathbb{C} \\ \text{is continuous}\end{array}\right)$$

$$= \lim_{k\to\infty} \big(\langle x, a_j \rangle - \langle x, a_j \rangle \big)$$

$$= 0.$$

So $\langle x - P(x), b \rangle = \lim_{n\to\infty} \langle x - P(x), b_n \rangle = \lim_{n\to\infty} 0 = 0.$

So $x - P(x) \in W^{\perp}$.