Metric and Hilbert Spaces 24.09.2014

Linear operators on Hilbert spaces

Theorem (Riesz representation Theorem).
Let H be a Hilbert space and
$H^* = B(H, \mathbb{C})$ the dual of H.

Then
$$\begin{array}{rl} H & \longrightarrow H^* \\ x & \longmapsto \varphi_x : H \longrightarrow \mathbb{C} \\ & \qquad y \longmapsto \langle y, x\rangle \end{array}$$
is a bijective linear transformation with
if $x \in X$ then $\|\varphi_x\| = \|x\|$.

HW Let H_1 and H_2 be Hilbert spaces and let
$T: H_1 \to H_2$ be a bounded linear operator.
Show that the adjoint of T is
$T^*: H_2 \longrightarrow H_1$ given by
if $x \in H_1$ and $y \in H_2$ then
$$\langle Tx, y\rangle_1 = \langle x, T^*y\rangle_2 .$$

Let H be a Hilbert space and let

$T: H \to H$ be a bounded linear operator.

(a) T is <u>self adjoint</u> if $T = T^*$.

(b) T is <u>positive</u> if $T = T^*$ <u>and</u>

if $x \in H$ then $\langle Tx, x\rangle \in \mathbb{R}_{\geq 0}$.

(c) T is <u>unitary</u> if $TT^* = T^*T = I$

(d) T is an <u>isometry</u> if T satisfies

if $x, y \in H$ then $\langle Tx, Ty\rangle_2 = \langle x, y\rangle_1$.

Let X be a normed vector space. Let

$$S = \{x \in X \mid \|x\| = 1\}$$

A bounded linear operator $T: X \to X$ is <u>compact</u>

if $\overline{T(S)}$ is compact.

<u>Theorem</u> 13.3 Let H be a Hilbert space.
Let $T: H \to H$ be a bounded self adjoint operator.
Then

$$\|T\| = \sup\{ |\langle Tx, x\rangle| \mid \|x\| = 1\}.$$

<u>Theorem</u> Let H be a Hilbert space and let

$T: H \to H$ be a nonzero selfadjoint compact operator.

(a) There exists $x \in H$ such that $\|x\| = 1$ and

if $u \in H$ and $\|u\| = 1$ then $|\langle Tu, u\rangle| \le |\langle Tx, x\rangle|$.

Then x is an eigenvector of T ~~and~~ with eigenvalue λ such that $|\lambda| = \|T\|$.

(b) There is an orthonormal basis of ~~eigenvectors~~ H consisting of eigenvectors of T.

(c) Let Λ be the set of eigenvalues of T and let $P(\mu): H \to H$ be the orthogonal projection onto X_μ, the subspace of eigenvectors with eigenvalue μ. Then

$$Tx = \sum_{\mu \in \Lambda} \mu P(\mu) x, \qquad \text{for } x \in H.$$

Examples of Hilbert spaces

(1) $\mathbb{C}^n$ with $\langle\,,\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by

$$\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}.$$

(2) ℓ^2 with $\langle\,,\rangle : \ell^2 \times \ell^2 \to \mathbb{C}$ given by

$$\langle (x_1, x_2, \dots), (y_1, y_2, \dots) \rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots = \sum_{i \in \mathbb{Z}_{>0}} x_i\overline{y_i}$$

(3) $$L^2[a,b] = \left\{ f : [a,b] \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is a limit of step functions} \\ \text{and } \|f\|_{L^2} < \infty \end{array} \right\}$$

with inner product $\langle\,,\rangle : L^2[a,b] \times L^2[a,b] \to \mathbb{C}$ given by

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\,dt.$$