Metric and Hilbert Spaces 23.09.2014

An <u>inner product space</u> is a vector space V over $\mathbb{C}$ with a function $V \times V \to \mathbb{C}$
$(v_1, v_2) \mapsto \langle v_1, v_2 \rangle$ such that

(a) If $v_1, v_2, v_3 \in V$ and $c_1, c_2 \in \mathbb{C}$ then

$$\langle c_1 v_1 + c_2 v_2, v_3 \rangle = c_1 \langle v_1, v_3 \rangle + c_2 \langle v_2, v_3 \rangle.$$

(b) If $v_1, v_2, v_3 \in V$ and $c_1, c_2 \in \mathbb{C}$ then

$$\langle v_3, c_1 v_1 + c_2 v_2 \rangle = \overline{c_1} \langle v_3, v_1 \rangle + \overline{c_2} \langle v_3, v_2 \rangle$$

(c) If $v_1, v_2 \in V$ then $\langle v_2, v_1 \rangle = \overline{\langle v_1, v_2 \rangle}$

(d) If $v \in V$ and $\langle v, v \rangle = 0$ then $v = 0$.

(e) If $v \in V$ then $\langle v, v \rangle \in \mathbb{R}_{\geq 0}$.

Let $(V, \langle\ \rangle)$ be an inner product space.
Define $\|\ \| : V \to \mathbb{R}_{\geq 0}$ by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

<u>HW</u> Show that $(V, \|\cdot\|)$ is a normed vector space.

A <u>Hilbert space</u> is an inner product space $(V, \langle\ \rangle)$ such that V is a complete metric space.

Orthogonal complements

Let $(V, \langle\ \rangle)$ be an inner product space and let $W \subseteq V$ be a subspace of V.

The orthogonal complement of W in V is

$$W^{\perp} = \{ v \in V \mid \text{if } w \in W \text{ then } \langle v, w \rangle = 0 \}.$$

Theorem If V is a Hilbertspace and W is closed then

$$V = W \oplus W^{\perp}.$$

Let V be an inner product space and let W be a subspace of V. An orthogonal projection onto W is a linear transformation

$P: V \to V$ such that

(a) if $v \in V$ then $P(v) \in W$,

(b) if $v \in V$ then $v - P(v) \in W^{\perp}$.

subtheorem Let V be a Hilbert space and let W be a subspace of V. There exists an orthogonal projection $P: V \to V$ onto W if and only if W is closed.

Orthogonality

HW Let $(V, \langle\,\rangle)$ be an inner product space. Show that

$$\begin{array}{rl} d: V & \to V^* \\ w & \mapsto \varphi_w : V \to \mathbb{C} \\ & \qquad v \mapsto \langle v, w\rangle \end{array}$$ is a linear transformation.

Let W be a subspace of V. Show that

$$W^\perp = \bigcap_{w \in W} \ker(\varphi_w).$$

Let V be a Hilbert space.

An orthonormal sequence in V is a sequence $a_1, a_2, a_3, \ldots$ in V such that

if $i, j \in \mathbb{Z}_{>0}$ then $\langle a_i, a_j\rangle = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$

HW Let $a_1, a_2, \ldots$ be an orthonormal sequence in V.

Let $W = \mathrm{span}\{a_1, a_2, \ldots\}$.

Show that

(a) If $v \in V$ then

$$\sum_{n \in \mathbb{Z}_{>0}} |\langle v, a_n\rangle|^2 \le \|v\|^2. \qquad \text{(Bessel's inequality)}$$

(b) $P: V \to V$ given by
$$P(v) = \sum_{n \in \mathbb{Z}_{>0}} \langle v, a_n \rangle a_n$$
is an orthogonal projection onto $\overline{W}$.

(c) If $\overline{W} = V$ then $\{a_1, a_2, a_3, \dots\}$ is a Schauder basis of V i.e., every $v \in V$ can be written uniquely as $v = \lambda_1 a_1 + \lambda_2 a_2 + \dots$.

HW (Fourier analysis).

Let $e_0, e_1, e_{-1}, e_2, e_{-2}, \dots$ in $L^2[0, 2\pi]$ be given by
$$e_m(t) = \frac{1}{\sqrt{2\pi}} e^{imt}.$$

Show that $e_0, e_1, e_{-1}, e_2, e_{-2}, \dots$ is an orthonormal basis of $L^2[0, 2\pi]$.

Gram-Schmidt

Let $v_1, v_2, \dots$ be a sequence of linearly independent vectors in V. Define
$$a_1 = \frac{v_1}{\|v_1\|} \quad \text{and} \quad a_{n+1} = \frac{v_{n+1} - \langle v_{n+1}, a_1 \rangle a_1 - \dots - \langle v_{n+1}, a_n \rangle a_n}{\|v_{n+1} - \langle v_{n+1}, a_1 \rangle a_1 - \dots - \langle v_{n+1}, a_n \rangle a_n\|}.$$

Then $a_1, a_2, \dots$ is an orthonormal sequence in V.