## Duals

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and let $V$ be a normed vector space over $\mathbb{F}$.

A <u>linear functional on $V$</u> is a linear operator $\varphi : V \to \mathbb{F}$. The <u>dual of $V$</u> is

$$V^* = B(V, \mathbb{F}) = \{ \varphi : V \to \mathbb{F} \mid \varphi \text{ is linear and } \|\varphi\| < \infty \}.$$

## Adjoints

Let $V$ and $W$ be normed vector spaces over $\mathbb{F}$.

Let $T : V \to W$ be a bounded linear operator.

The <u>adjoint of $T$</u> is the function

$$T^* : W^* \to V^* \text{ given by } (T\psi)(v) = (\psi \circ T)(v).$$

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "\psi"] \\ & \mathbb{F} \end{tikzcd}$$

<u>HW</u> Show that $T^* : W^* \to V^*$ is a linear operator.

<u>HW</u> Show that $\|T^*\| = \|T\|$.

HW Show that

$$ev: V \longrightarrow V^{**}$$
$$x \longmapsto ev_x : V \longrightarrow \mathbb{F}$$
$$\varphi \longmapsto \varphi(x)$$

is an injective linear transformation and $\|x\| = \|ev_x\|$.

A normed vector space $V$ is reflexive if $ev: V \to V^{**}$ is a bijection.

HW Show that if $V$ and $W$ are reflexive then $T^{**} = T$.

HW Let $p \in \mathbb{R}_{>1}$ and let $q \in \mathbb{R}_{>1}$ be given by $\frac{1}{p} + \frac{1}{q} = 1$. Show that $(\ell^p)^* = \ell^q$.

HW (a) Show that if $p \in \mathbb{R}_{>1}$ then $\ell^p$ is reflexive.

(b) Show that $\ell^1$ is not reflexive

(c) Show that $\ell^\infty$ is not reflexive.

HW (a) Show that $(\ell^1)^* = \ell^\infty$

(b) Show that $(\ell^\infty)^* \neq \ell^1$.

<u>Theorem</u> <u>Riesz representation Theorem</u>

Let $H$ be a Hilbert space. Then

$$\begin{aligned} H &\longrightarrow H^* \\ a &\longmapsto \varphi_a : H \to \mathbb{F} \\ &\qquad\quad x \longmapsto \langle x, a \rangle \end{aligned}$$ is a vector space isomorphism.

<u>HW</u> Let $(H_1, \langle\ \rangle_{H_1})$ and $(H_2, \langle\ \rangle_{H_2})$ be Hilbert spaces.
Let $T: H_1 \to H_2$ be a bounded linear operator.
Show that the adjoint of $T$ is the function
$T^*: H_2 \to H_1$ given by $\langle T^*y, x\rangle_{H_1} = \langle y, Tx\rangle_{H_2}$

<u>Finite dimensional vector spaces</u>

Let $V$ and $W$ be finite dimensional vector spaces over $\mathbb{F}$. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $V$ and let $\{w_1, w_2, \ldots, w_m\}$ be a basis of $W$.
The <u>dual</u> <u>basis</u> to $\{v_1, v_2, \ldots, v_n\}$ is the basis $\{v^1, v^2, \ldots, v^n\}$ of $V^*$ given by

$$v^i(v_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \qquad \text{for } i, j \in \{1, 2, \ldots n\}.$$

Let $\{w^1, w^2, \ldots, w^m\}$ be the dual basis to $\{w_1, w_2, \ldots w_m\}$.
$\{w^1, w^2, \ldots, w^m\}$ is a basis of $W^*$.

HW Let $T: V \to W$ be a linear operator and let

$T_{ij} \in \mathbb{F}$ be given by

$$Tv_i = \sum_{j=1}^{m} T_{ji}\, w_j .$$

Show that

$$T^* w^j = \sum_{i=1}^{n} T_{ji}\, v^i$$

by evaluating each side at $v_k$.

HW If $\langle , \rangle$ is an inner product on $V$ and

$\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of $V$

with respect to $\langle , \rangle$ and

$T: V \to V$ is a linear operator

and

$$Tv_i = \sum_{j=1}^{n} T_{ji}\, v_j \qquad \text{then} \qquad T^* v_i = \sum_{i=1}^{n} \overline{T_{ij}}\, v_j .$$