Metric and Hilbert spaces 17.09.2014

Let V and W be normed vector spaces over $\mathbb{F}$, where $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$.

A <u>bounded</u> <u>linear</u> <u>operator</u> <u>from V to W</u> is a linear operator $T: V \to W$ such that there exists $C \in \mathbb{R}_{>0}$ such that if $u \in V$ then $\|Tu\| \le C\|u\|$.

The <u>norm of T</u> is the minimal C that works.

<u>Examples</u>

(1) Let V be a normed vector space and let

$$\begin{array}{rcl} 1: V & \to & V \\ x & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} 0: V & \to & V \\ x & \mapsto & 0 \end{array}$$

Then $\|1\|=1$ and $\|0\|=0$.

(2) Let $V=\mathbb{R}^n$ or $V=\mathbb{C}^n$ and let $T: V \to W$ be a linear operator. Then the function

$$\begin{array}{rcl} \Phi: V & \to & \mathbb{R}_{\ge 0} \\ x & \mapsto & \|Tx\| \end{array}$$ is continuous (Really? Why?)

Let

$$S^{n-1} = \{x \in V \mid \|x\|=1\}$$

Since S^{n-1} is closed and bounded, S^{n-1} is compact.

Since Φ is continuous, $\Phi(S^{n-1})$ is compact.

So $\{\|Tx\| \mid \|x\|=1\}$ is closed and bounded in $\mathbb{R}_{\ge 0}$.

So $\|T\| = \sup(\Phi(S^{n-1}))$.

(3) Let $C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$,

with norm given by

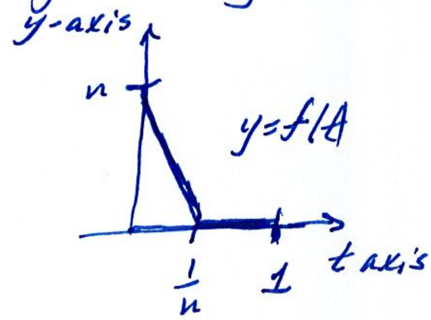
$$\|f\| = \int_0^1 |f(t)|\,dt.$$

Let $T : C[0,1] \to \mathbb{R}$ be given by

$$Tf = ev_0(f) = f(0).$$

Let $f_1, f_2, \ldots$ be the sequence in $C[0,1]$ given by

$$f_n(t) = \begin{cases} -n^2t + n, & \text{for } t \in [0, \frac{1}{n}] \\ 0, & \text{for } t \in [\frac{1}{n}, 1] \end{cases}$$



Then

$$\|f_n\| = \frac{n(\frac{1}{n})}{2} = \frac{1}{2} \quad \text{(the area under } y = f_n(t)\text{), and}$$

$$\lim_{n\to\infty} Tf_n = \infty \quad \text{in } \mathbb{R}_{\geq 0} \cup \{\infty\}$$

since $Tf_1 = 1$, $Tf_2 = 2$, $Tf_3 = 3, \ldots$

So there does not exist $C \in \mathbb{R}_{\geq 0}$ such that

if $f \in C[0,1]$ then $\|Tf\| \leq C\|f\|$.

(4) Let $\lambda_1, \lambda_2, \ldots$ be a bounded sequence in $\mathbb{R}$.
Define $T: \ell^\infty \to \ell^\infty$ by

$$T(a_1, a_2, \ldots) = (\lambda_1 a_1, \lambda_2 a_2, \ldots).$$

The norm on ℓ^∞ is

$$\|(a_1, a_2, \ldots)\| = \sup\{|a_1|, |a_2|, \ldots\}.$$

A basis of ℓ^∞ is $\{e_1, e_2, e_3, \ldots\}$ where

$$e_i = (0, 0, \ldots, 0, \overset{i^{th}\,spot}{1}, 0, 0, \ldots).$$

Then

$$Te_i = (0, \ldots, 0, \lambda_i, 0, 0, \ldots) \text{ and } \|Te_i\| = |\lambda_i| = |\lambda_i|\,\|e_i\|.$$

So $\|T\| \geq \sup\{|\lambda_1|, |\lambda_2|, \ldots\}$.

Let $x = (x_1, x_2, \ldots)$ in ℓ^∞. Then

$$\|Tx\| = \|(\lambda_1 x_1, \lambda_2 x_2, \ldots)\| = \sup\{|\lambda_1||x_1|, |\lambda_2||x_2|, \ldots\}$$

$$\leq \sup\{|\lambda_1|, |\lambda_2|, \ldots\} \cdot \sup\{|x_1|, |x_2|, \ldots\} = C\|x\|$$

where $C = \sup\{|\lambda_1|, |\lambda_2|, \ldots\}$.

So $\|T\| \leq C$.

So $\|T\| = \sup\{|\lambda_1|, |\lambda_2|, \ldots\}$.

(5) Let $C[a,b] = \{f: [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$
with the sup norm

$$\|f\| = \sup\{|f(t)| \mid t \in [a,b]\}.$$

Let $T: C[a,b] \to \mathbb{R}$ be given by

$$Tf = \int_a^b f(t)\,dt.$$

If $x: [a,b] \to \mathbb{R}$ is the function given by $x(t) = 1$ then

$$Tx = \int_a^b dt = b - a \quad \text{and} \quad \|Tx\| = (b-a) = (b-a)\|x\|$$

since $\|x\| = 1$. So $\|T\| \geq b - a$.

If $f \in C[a,b]$ then

$$\|Tf\| = \left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt \leq \|f\|(b-a).$$

So $\|T\| \leq b - a$. Thus $\|T\| = b - a$.

(6) <u>Integral operators</u>. Let

$$C[a,b] = \{ f: [a,b] \to \mathbb{C} \mid f \text{ is continuous} \}$$

with norm given by

$$\|f\| = \sup\{ |f(t)| \mid t \in [a,b] \}.$$

Let $K: [a,b] \times [a,b] \to \mathbb{C}$ be a continuous function.

Define $T: C[a,b] \to C[a,b]$ by

$$(Tf)(t) = \int_a^b K(t,s) f(s)\,ds$$

(generalised matrix multiplication!).

To show: (a) If $f \in C[a,b]$ then $Tf \in C[a,b]$.

(b) T is a bounded linear operator.

(a) Assume $f \in C[a,b]$.

To show: Tf is continuous.

In fact we will show: Tf is uniformly continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $t, t' \in [a,b]$ and $|t-t'| < \delta$ then

$$|Tf(t) - Tf(t')| < \varepsilon.$$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Since $[a,b] \times [a,b]$ is compact and K is continuous then K is uniformly continuous.

Let $\delta \in \mathbb{R}_{>0}$ be such that if $s, s', t, t' \in [a,b]$ and

$$d((s,t),(s',t')) < \delta \text{ then } |K(s,t) - K(s',t')| < \frac{\varepsilon}{(b-a)\|f\|}.$$

To show: If $t, t' \in [a,b]$ and $|t-t'| < \delta$ then

$$|Tf(t) - Tf(t')| < \varepsilon.$$

Assume $t, t' \in [a,b]$ and $|t-t'| < \delta$.

To show: $|Tf(t) - Tf(t')| < \varepsilon$.

$$|Tf(t)-Tf(t')| = \left|\int_a^b (K(s,t)-K(s,t'))f(s)\,ds\right|$$

$$\leq \int_a^b |K(s,t)-K(s,t')|\cdot|f(s)|\,ds$$

$$< \frac{\varepsilon}{(b-a)\|f\|}\cdot(b-a)\|f\| = \varepsilon.$$

So Tf is uniformly continuous.

So Tf is continuous and so $Tf \in C[a,b]$.

(b) To show: T is a bounded linear operator.

To show: There exists $C \in \mathbb{R}_{>0}$ such that if $f \in C[a,b]$ then $\|Tf\| \leq C\|f\|$.

Since $[a,b]\times[a,b]$ is compact and K is continuous $K([a,b]\times[a,b])$ is compact and $K([a,b]\times[a,b])$ is bounded.

Let

$$C = \sup(\{|K(s,t)| \mid s,t \in [a,b]\})$$

To show: If $f \in C[a,b]$ then $\|Tf\| \leq C\|f\|$.

Assume $f \in C[a,b]$.

Then $|Tf(t)| \leq \int_a^b |K(s,t)|\cdot|f(s)|\,ds \leq (b-a)C\|f\|$

So $\|Tf\| = \sup\{|Tf(t)| \mid t \in [a,b]\} \leq (b-a)C\|f\|$.

So $\|T\| \leq (b-a)C$. So $\|T\|$ is bounded.