Let V and W be vector spaces over $\mathbb{F}$.

A <u>linear operator</u> is a function $T: V \to W$ such that

(a) if $v_1, v_2 \in V$ then $T(v_1 + v_2) = T(v_1) + T(v_2)$,

(b) if $v \in V$ and $\lambda \in \mathbb{F}$ then $T(\lambda v) = \lambda T(v)$.

Let V and W be normed vector spaces over $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

A <u>bounded linear operator from V to W</u> is a linear operator $T: V \to W$ such that

there exists $c \in \mathbb{R}_{>0}$ such that

if $x \in V$ then $\|Tx\| \le c\|x\|$.

The <u>operator norm</u> of a bounded linear operator $T: V \to W$ is

$$\|T\| = \sup\left\{ \frac{\|Tx\|}{\|x\|} \;\middle|\; x \in V \text{ and } x \neq 0 \right\} \quad (*)$$

<u>HW</u> Let V and W be normed vector spaces. Show that

$$B(V, W) = \{\text{bounded linear operators } T: V \to W\}$$

with norm $\|\cdot\|: B(V, W) \to \mathbb{R}_{\ge 0}$ given by $(*)$ is a normed vector space.

Theorem Let V and W be normed vector spaces. If W is a Banach space then $B(V,W)$ is a Banach space

Proof Assume W is a Banach space.

To show: $B(V,W)$ is complete

To show: If $T_1, T_2, \ldots$ is a Cauchy sequence in $B(V,W)$ then $T_1, T_2, \ldots$ converges to $T \in B(V,W)$.

Assume $T_1, T_2, \ldots$ is a Cauchy sequence in $B(V,W)$.

To show: There exists $T \in B(V,W)$ such that $\lim_{n \to \infty} \|T_n - T\| = 0$.

Let $T: V \to W$ be given by

$$T(x) = \lim_{n \to \infty} T_n(x).$$

To show: (a) $\lim_{n \to \infty} T_n(x)$ exists in W.

(b) $T \in B(V,W)$

(c) $\lim_{n \to \infty} \|T_n - T\| = 0.$

From here it is possible to follow the proof in the Rubinstein notes.

<u>Theorem</u> Let V and W be normed vector spaces and let $T: V \to W$ be a linear operator.

Then

(a) $T \in B(V, W)$ if and only if T is continuous.

(b) T is continuous if and only if T is uniformly continuous.

<u>Proof</u> To show: (a) If $T \in B(V,W)$ then T is uniformly continuous

(b) If T is uniformly continuous then T is continuous.

(c) If T is continuous then $T \in B(V, W)$.

(b) was ~~proved by~~ ~~[illegible]~~ a direct consequence of the way that we set up the definitions of 'uniformly continuous' and 'continuous'.

(a) Assume $T \in B(V, W)$.

To show: T is uniformly continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in V$ and $d(x,y) < \delta$ then $d(Tx, Ty) < \varepsilon$.

This is a consequence of the computation

$$d(Tx, Ty) = \|Tx - Ty\| = \|T(x-y)\| \le \|T\| \, \|x-y\| = \|T\| \, d(x,y).$$

(c) To show: If T is continuous then $T \in B(V,W)$.

Assume T is continuous.

To show: T is bounded.

To show: There exists $C \in \mathbb{R}_{>0}$ such that

if $u \in V$ then $\|Tu\| \le C\|u\|$.

Since T is continuous, T is continuous at 0.

So, there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x \in V$ and $\|x\| < \delta$ then $\|Tx\| < 1$.

Let $C = \frac{2}{\delta}$.

To show: If $u \in V$ then $\|Tu\| \le C\|u\|$.

Assume $u \in V$.

Let $x = \frac{\delta}{2}\frac{u}{\|u\|}$ so that $\|x\| < \frac{\delta}{2}$

Then

$$1 > \|Tx\| = \left\|T\left(\frac{\delta}{2}\frac{u}{\|u\|}\right)\right\| = \frac{\delta}{2\|u\|}\|Tu\|.$$

So $\|Tu\| < \frac{2}{\delta}\|u\| = C\|u\|$.

So T is bounded //.