# Lecture 20: Metric and Hilbert spaces

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Let $(X, d)$ be a metric space.

A <u>contraction</u> <u>mapping</u> is a function $f: X \to X$ such that there exists $\alpha \in (0,1)$ such that

if $x, y \in X$ then $d(f(x), f(y)) \le \alpha\, d(x,y)$.

A fixed point of $f: X \to X$ is an element $x \in X$ such that $f(x) = x$.

<u>Theorem</u>: <u>Banach fixed point theorem</u>

Let $(X, d)$ be a complete metric space and let $f: X \to X$ be a contraction mapping. Let $x \in X$ and let $x_1, x_2, \ldots$ be the sequence

$$x_1 = f(x),\quad x_2 = f(f(x)),\quad x_3 = f(f(f(x))),\ \ldots$$

Then $x_1, x_2, \ldots$ converges and

$p = \lim_{n\to\infty} x_n$ is the unique fixed point of $f$.

<u>Proof</u>: To show: (a) $x_1, x_2, \ldots$ is a Cauchy sequence

(b) $p = \lim_{n\to\infty} x_n$ exists

(c) $f(p) = p$

(d) If $q$ is a fixed point of $f$ then $q = p$.

(a) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(x_m, x_n) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

Let $N$ be the smallest integer in $\mathbb{Z}_{>0}$ such that $\dfrac{\alpha^N d(f(x), x)}{1-\alpha} < \varepsilon$ $\left(\text{so } \alpha^N < \dfrac{\varepsilon(1-\alpha)}{d(f(x), x)}\right)$.

To show: If $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(x_m, x_n) < \varepsilon$.

Assume $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$. Assume $n < m$.

To show: $d(x_m, x_n) < \varepsilon$.

Since

$$d(x_2, x_1) = d(f^2(x), f(x)) \le \alpha\, d(f(x), x),$$

$$d(x_3, x_2) = d(f^3(x), f^2(x)) \le \alpha\, d(f^2(x), f(x)) \le \alpha^2 d(f(x), x),$$

$$d(x_4, x_3) = d(f^4(x), f^3(x)) \le \alpha\, d(f^3(x), f^2(x)) \le \alpha^3 d(f(x), x),$$

$\ldots$

then

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\le \alpha^n d(f(x), x) + \alpha^{n+1} d(f(x), x) + \cdots + \alpha^{m-1} d(f(x), x) \\ &= (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1})\, d(f(x), x) \\ &\le \alpha^n (1 + \alpha + \alpha^2 + \cdots)\, d(f(x), x) = \frac{\alpha^n}{1-\alpha} d(f(x), x) < \varepsilon. \end{aligned}$$

So $x_1, x_2, \ldots$ is a Cauchy sequence in $X$.

(b) Since $(X,d)$ is complete and $x_1, x_2, \ldots$ is a Cauchy sequence in $X$ then $x_1, x_2, \ldots$ converges in $X$. So there exists $p \in X$ with $p = \lim_{n\to\infty} x_n$.

(c) To show: $f(p) = p$.

To show: $d(f(p), p) = 0$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then $d(f(p), p) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Since $\lim_{n\to\infty} x_n = p$, there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n \geq N$ then $d(x_n, p) < \frac{\varepsilon}{2}$.

Then

$$\begin{aligned} d(f(p), p) &\leq d(f(p), x_{N+1}) + d(x_{N+1}, p) \\ &\leq \alpha\, d(p, x_N) + d(x_{N+1}, p) \quad (\text{since } x_{N+1} = f(x_N)) \\ &< \alpha \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

So $d(f(p), p) = 0$.

So $f(p) = p$.

(d) To show: If $q$ is a fixed point of $f$ then $q=p$.

To show: If $q \in X$ and $f(q)=q$ then $q=p$.

Assume $q \in X$ and $f(q)=q$.

To show: $q=p$.

To show: $d(q,p)=0$.

$d(q,p) = d(f(q), f(p))$ (since $q=f(q)$ and $p=f(p)$)

$\le \alpha\, d(q,p)$ (since $f$ is contractive).

So $(1-\alpha)\, d(q,p) \le 0$.

Since $(1-\alpha)\, d(q,p) \ge 0$ and $(1-\alpha)\, d(q,p) \le 0$ then $(1-\alpha)\, d(q,p) = 0$.

Since $\alpha \in (0,1)$ then $(1-\alpha) \ne 0$, and so $d(q,p)=0$.

So $p=q$. //.