Metric and Hilbert Spaces: Lecture 1 29 July 2014

Information

(1) Google: Arun Ram

(2) Contact and availability

(3) SSLC Representatives

(4) Scribe for

HWs, Vocabulary and Examples.

(5) Books: Rubinstein notes and online Notes.

(6) Schedule - Times away.

(7) Homework and Exams.

(8) Proof Machine.

BIG IDEA of the course: CONVERGENCE

Definition: A sequence $(x_1, x_2, x_3, \ldots)$ converges to x if $(x_1, x_2, \ldots)$ satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(x_n, x) < \varepsilon$.

Write

$\lim_{n \to \infty} x_n = x$ if $(x_1, x_2, \ldots)$ converges to x.

Rubinstein writes:

"Definition 2.8." The sequence $\{x_n\}$ is said to <u>converge</u> to a point x in X, if for every $\varepsilon > 0$ there exists a positive integer k such that

$$d(x_n, x) < x \quad \text{for all } n \geq k.$$

In this case we write

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \longrightarrow x$$

The point x is called the <u>limit</u> of $\{x_n\}$.

Examples

(1) $\mathbb{R}^n = \{ x = (x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R} \}$

$= \{ x : [1,n]_{\mathbb{Z}} \to \mathbb{R} \} = \{$functions from $\{1, 2, \ldots, n\}$ to $\mathbb{R}\}$.

Possible norms on $\mathbb{R}^n$:

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_p|^p)^{1/p}$$

$$\|x\|_\infty = \sup\{ |x_1|, |x_2|, \ldots, |x_n| \}$$

(2) $\mathbb{R}^\infty = \{ x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \} = \{$sequences $x_1, x_2, \ldots$ in $\mathbb{R}\}$

$= \{ x : \mathbb{Z}_{>0} \to \mathbb{R} \} = \{$functions from $\{1, 2, \ldots\}$ to $\mathbb{R}\}$

Possible norms on $\mathbb{R}^\infty$:

$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}$ gives ℓ^2,

$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$ gives ℓ^p,

$\|x\|_\infty = \cancel{\sum_{i=1}^{\infty}} \sup\{ |x_1|, |x_2|, \ldots \}$ gives ℓ^∞

(3) Consider $\{ f : [0,1] \to \mathbb{R} \}$

or $\{ f : X \to \mathbb{R} \}$.

Can we put norms on these to get $L^2(X)$, $L^p(X)$, $L^\infty(X)$?