Lecture 19: Metric and Hilbert Spaces 28 August 2014

A Hausdorff topological space is a topological space $(X, \mathcal{T})$ such that

if $x_1, x_2 \in X$ and $x_1 \neq x_2$ then there exist $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{T}$ such that $x_1 \in \mathcal{U}_1$, $x_2 \in \mathcal{U}_2$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$.

A normal topological space is a topological space $(X, \mathcal{T})$ such that

if C_1, C_2 are closed sets in X and $C_1 \cap C_2 = \phi$ then there exist $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{T}$ such that $C_1 \subseteq \mathcal{U}_1$, $C_2 \subseteq \mathcal{U}_2$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$.

HW If X is a Hausdorff topological space and $A \subseteq X$ then A is cover compact $\Rightarrow$ A is closed

HW If X is a compact Hausdorff topological space then X is normal.

A topological space $(X, \mathcal{T})$ is path connected if X satisfies:

if $p, q \in X$ then there exists a continuous function $f: [0,1] \to X$ with $f(0) = p$ and $f(1) = q$.

HW Show that if X is path connected then X is connected.

HW Show that the graph of $f(x) = \begin{cases} \sin(\frac{1}{x}), & \text{if } x \in (0,1] \\ 0, & \text{if } x = 0 \end{cases}$ is a connected set which is not path connected.

One point compactification

A topological space ~~X~~ is <u>locally compact</u> if X is Hausdorff and

if $x \in X$ then there exists $N \in \mathcal{N}(x)$ such that

N is compact.

<u>HW</u> Show that $\mathbb{R}$ is locally compact but $\mathbb{R}$ is not compact.

Let X be locally compact.

The <u>one point compactification</u> of X is

$X' = X \cup \{\omega\}$ with topology

$\mathcal{T}' = \mathcal{T} \cup \{(X - K) \cup \{\omega\} \mid K \subseteq X \text{ and } K \text{ is compact}\}$

Compactness and closedness

HW Let (X,d) be a metric space and let $Y \subseteq X$.

(a) Show that if X is compact and Y is closed then Y is compact.

(b) Show that if Y is compact then Y is closed.

Example Closed and bounded $\not\Rightarrow$ compact.

Let $X = C([0,1];\mathbb{R})$ with metric given by

$$d(f,g) = \sup\{|f(x)-g(x)| \mid x \in [0,1]\}.$$

Let $E = \overline{B(0,1)} = \{f \in X \mid d(f,0) \le 1\}.$

Since d is continuous then E is closed

Since $E \subseteq B(0,2)$ then E is bounded.

Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^n$.

The pointwise limit of $f_1, f_2, \ldots$ is $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if } x \ne 1, \\ 1, & \text{if } x = 1. \end{cases}$$

Since $\|f_n - f\| = 1$ for $n \in \mathbb{Z}_{>0}$,

$$\lim_{n\to\infty} d(f_n, f) = \lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} 1 = 1.$$

So $f_1, f_2, \ldots$ does not have a convergent subsequence.

Examples of completions

(1) The completion of $\mathbb{Q}$ is $\mathbb{R}$.

(2) The completion of $\mathbb{Q}$ is $\mathbb{Q}_p$

(3) The completion of $\mathbb{C}[x]$ is $\mathbb{C}[[x]]$

(4) The completion of $\mathbb{C}(x)$ is $\mathbb{C}((x))$.