Lecture 18: Metric and Hilbert spaces 27 August 2014

Compactness

Let (X,d) be a metric space. Let $A \subseteq X$

The set A is sequentially compact if every sequence in A has a cluster point in A.

The set A is Cauchy compact, or complete, if every Cauchy sequence in A has a cluster point in A.

The set A is ball compact if A satisfies if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_N \in X$ such that

$$A \subseteq B(x_1,\varepsilon) \cup B(x_2,\varepsilon) \cup \cdots \cup B(x_N,\varepsilon).$$

The set A is cover compact if A satisfies if $\mathcal{S} \subseteq \mathcal{T}$ and $A \subseteq \bigcup_{U \in \mathcal{S}} U$ then there exists $N \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_N \in \mathcal{S}$ such that

$$A \subseteq U_1 \cup U_2 \cup \cdots \cup U_N.$$

(in English: every open cover has a finite subcover).

Synonyms for ball compact are precompact and totally bounded.

<u>Theorem</u> Let (X,d) be a metric space and let $A \subseteq X$.

(a) If A is cover compact then A is sequentially compact.

(b) If A is sequentially compact then A is cover compact.

(c) If A is sequentially compact then A is Cauchy compact.

(d) If A is cover compact then A is ball compact.

(e) If A is ball compact then A is bounded.

(f) If A is Cauchy compact then A is closed.

$$\begin{tikzcd}
\text{cover compact} \arrow[r, Rightarrow, "(d)"] \arrow[d, Rightarrow, shift right, "(a)"'] & \text{ball compact} \arrow[r, Rightarrow, "(e)"] & \text{bounded} \\
\text{sequentially compact} \arrow[u, Rightarrow, shift right, "(b)"'] \arrow[r, Rightarrow, "(c)"] & \text{Cauchy compact} \arrow[r, Rightarrow, "(f)"] & \text{closed}
\end{tikzcd}$$

<u>HW</u> Let $A = (0,1) \subseteq \mathbb{R}$ with the standard metric on $\mathbb{R}$.
Show that A is ball compact and not closed but not cover compact (and not Cauchy compact).

<u>HW</u> Let $X = \mathbb{R}$ with metric given by $d(x,y) = \min\{|x-y|, 1\}$

Show that X is bounded but not ball compact.

<u>HW</u> Let $A = \mathbb{R}_{(0,1)}$ with metric $d(x,y) = |x-y|$.

Show that A is closed but A is not Cauchy compact.

<u>HW</u> Show that $\mathbb{R}$ with the standard metric is and not bounded ~~and not bounded~~ Cauchy compact but not cover compact (and not ball compact).

<u>Theorem</u> Let (X,d) be a metric space and let $A \subseteq X$.

If A is ball compact and Cauchy compact then A is cover compact.

<u>Theorem</u> Let $A \subseteq \mathbb{R}^n$ with the standard metric on $\mathbb{R}^n$.

If A is closed and bounded then

A is cover compact.

<u>Proof</u> Assume $A \subseteq \mathbb{R}^n$. and A is closed and bounded.

(a) Since $\mathbb{R}^n$ is complete, $A \subseteq \mathbb{R}^n$ and A is closed then A is complete. So A is Cauchy compact.

(b) Since $A \subseteq \mathbb{R}^n$ and A is bounded then A is ball compact.

Since A is Cauchy compact and ball compact.

So A is cover compact.

(Note: (b) follows since $A \subseteq B(0,c)$ for some $c \in \mathbb{R}_{>0}$ and $B(0,c)$ is ball compact in $\mathbb{R}^n$. (since $B(0,c)$ has finite volume))

<u>HW</u> Show that a closed subset of a Cauchy compact set is Cauchy compact.

<u>HW</u> Show that a bounded subset of a ball compact set is ball compact.

<u>HW</u> Show that a closed subset of a cover compact set is cover compact.

Example $\ell^2 = \{\vec{x} = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ with } \|\vec{x}\|_2 < \infty\}$

where

$$\|\vec{x}\|_2 = \Big(\sum_{i \in \mathbb{Z}_{>0}} x_i^2\Big)^{\frac{1}{2}} \quad \text{for} \quad \vec{x} = (x_1, x_2, \ldots).$$

Then let

$$e_1 = (1, 0, 0, \ldots)$$
$$e_2 = (0, 1, 0, \ldots)$$
$$e_3 = (0, 0, 1, 0, \ldots)$$
$$\vdots$$

Then $e_1, e_2, e_3, \ldots$ is a sequence in ℓ^2

$\{e_1, e_2, e_3, \ldots\}$ is bounded since $d(e_i, e_j) = \sqrt{1+1} = \sqrt{2}$.

The set $\{e_1, e_2, e_3, \ldots\}$ is not ball compact.

Connectedness, compactness and the Mean value theorem

Theorem Let X, Y be topological spaces and let $f: X \to Y$ be a continuous function. Let $E \subseteq X$.

(a) If E is cover compact then $f(E)$ is cover compact.

(b) If E is connected then $f(E)$ is connected.

Corollary Let $f: X \to \mathbb{R}$ be a continuous function where X is a compact metric space. Then f attains a maximum and minimum value, i.e. there exist

$a \in X$ such that $f(a) = \inf\{f(x) \mid x \in X\}$

and

$b \in X$ such that $f(b) = \sup\{f(x) \mid x \in X\}$.

Proposition Let $A \subseteq \mathbb{R}$.

(a) A is connected if and only if A is an interval.

(b) A is compact and connected then A is a closed bounded interval.

Rolle's theorem $f: [a,b] \to \mathbb{R}$ with $f(a) = f(b)$

The Mean value theorem $f: [a,b] \to \mathbb{R}$. There exists $c \in [a,b]$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Sketches of proofs

(a) cover compact $\Rightarrow$ sequentially compact.

To show: not sequentially compact $\Rightarrow$ not cover compact

Let $a_1, a_2, \ldots$ be a sequence in A with no cluster point.

For each $a \in A$ let $\varepsilon_a \in \mathbb{R}_{>0}$ be such that

$$B(a, \varepsilon_a) \cap \{a_1, a_2, \ldots\} \text{ is finite.}$$

Then $\{B(a, \varepsilon_a) \mid a \in A\}$ is an open cover of A with no finite subcover.

So A is not cover compact.

(b) sequentially compact $\Rightarrow$ cover compact

To show: not cover compact $\Rightarrow$ not sequentially compact

Let $\mathcal{S}$ be a cover of A with no finite subcover.

Let $a_1 \in A$ and $S_1 \in \mathcal{S}$ with $a_1 \in S_1$.

Let $a_2 \in$ ~~A and S~~ $A \setminus S_1$ and $S_2 \in \mathcal{S}$ with $a_2 \in S_2$

Let $a_3 \in A \setminus (S_1 \cup S_2)$ and $S_3 \in \mathcal{S}$ with $a_3 \in S_3$

$\vdots$

Then $a_1, a_2, a_3, \ldots$ is a sequence in A with no cluster point.

(a cluster point a would have an $S \in \mathcal{S}$ with $a \in S$, and so S is a neighborhood of a and would contain all but a finite number of a_i and so $S_1 \cup S_2 \cup \cdots \cup S_N \cup S$ would be a finite cover??).