Lecture 15: Metric and Hilbert spaces 21 August 2014

(ab) To show: If $\vec{x},\vec{y}\in\hat{X}$ then $\hat{d}(\vec{x},\vec{y})=\hat{d}(\vec{y},\vec{x})$

Assume $\vec{x},\vec{y}\in\hat{X}$ with $\vec{x}=(x_1,x_2,\dots)$ and $\vec{y}=(y_1,y_2,\dots)$.

Since $d(x_n,y_n)=d(y_n,x_n)$,

$$\hat{d}(\vec{x},\vec{y})=\lim_{n\to\infty}d(x_n,y_n)=\lim_{n\to\infty}d(y_n,x_n)=\hat{d}(\vec{y},\vec{x}).$$

(ac) To show: If $\vec{x}\in\hat{X}$ then $\hat{d}(\vec{x},\vec{x})=0$.

Assume $\vec{x}\in\hat{X}$.

To show: $\hat{d}(\vec{x},\vec{x})=0$.

Since $d(x_n,x_n)=0$,

$$\hat{d}(\vec{x},\vec{x})=\lim_{n\to\infty}d(x_n,x_n)=\lim_{n\to\infty}0=0.$$

(ad) To show: If $\vec{x},\vec{y}\in\hat{X}$ and $\hat{d}(\vec{x},\vec{y})=0$ then $\vec{x}=\vec{y}$.

Assume $\vec{x},\vec{y}\in\hat{X}$ and $\hat{d}(\vec{x},\vec{y})=0$.

To show: $\vec{x}=\vec{y}$.

This is a consequence of the definition of $\hat{X}$ which requires that $\vec{x}=\vec{y}$ if $\hat{d}(\hat{x},\hat{y})=0$.

(ae) If $\vec{x},\vec{y},\vec{z}\in\hat{X}$ then $\hat{d}(\vec{x},\vec{y})\le\hat{d}(\vec{x},\vec{z})+\hat{d}(\vec{z},\vec{y})$.

Assume $\vec{x},\vec{y},\vec{z}\in\hat{X}$.

To show: $\hat{d}(\vec{x},\vec{y})\le\hat{d}(\vec{x},\vec{z})+\hat{d}(\vec{z},\vec{y})$.

$$\hat{d}(\vec{x},\vec{y})=\lim_{n\to\infty}d(x_n,y_n)\le\lim_{n\to\infty}\left(d(x_n,z_n)+d(z_n,y_n)\right)$$

$$= \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n)$$

$$= \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y}),$$

where the next to last ~~in~~equality follows from the continuity of addition in $\mathbb{R}_{\geq 0}$.

(d) To show: $\overline{\varphi(X)} = \hat{X}$

To show: If $\vec{z} \in \hat{X}$ then there exists a sequence $\vec{x}_1, \vec{x}_2, \ldots$ in $\varphi(X)$ such that $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

Let $\vec{z} = (z_1, z_2, \ldots) \in \hat{X}$.

To show: There exists $\vec{x}_1, \vec{x}_2, \ldots$ in $\varphi(X)$ with $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

Let
$$\vec{x}_1 = (z_1, z_1, z_1, z_1, \ldots) = \varphi(z_1),$$
$$\vec{x}_2 = (z_2, z_2, z_2, z_2, \ldots) = \varphi(z_2),$$
$$\vec{x}_3 = (z_3, z_3, z_3, \ldots) = \varphi(z_3), \quad \ldots$$

Then $\vec{x}_1, \vec{x}_2, \ldots$ is the sequence $\varphi(z_1), \varphi(z_2), \ldots$ in $\varphi(X)$.

To show: $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

To show: $\lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) = 0$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $\hat{d}(\vec{x}_n, \vec{z}) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

Let $N \in \mathbb{Z}_{>0}$ be such that if $r, s \in \mathbb{Z}_{>0}$ and $r > N$ and $s > N$ then $d(z_r, z_s) < \varepsilon/2$.

To show: If $n \in \mathbb{Z}_{>0}$ and $n > N$ then $\hat{d}(\vec{x}_n, \vec{z}) < \varepsilon$.

Assume $n \in \mathbb{Z}_{>0}$ and $n > N$.

To show: $\hat{d}(\vec{x}_n, \vec{z}) < \varepsilon$.

To show: $\lim\limits_{k\to\infty} d(z_n, z_k) < \varepsilon$.

$$\lim_{k\to\infty} d(z_n, z_k) \le \frac{\varepsilon}{2} < \varepsilon,$$

since $d(z_n, z_k) < \frac{\varepsilon}{2}$ for $k > N$.

So $\lim\limits_{n\to\infty} \vec{x}_n = \vec{z}$.

So $\overline{\varphi(X)} = \hat{X}$.

(b) To show: $(\hat{X}, \hat{d})$ is complete

To show: If $\vec{x}_1, \vec{x}_2, \ldots$ is a Cauchy sequence in $\hat{X}$ then $\vec{x}_1, \vec{x}_2, \ldots$ converges.

Assume

$$\vec{x}_1 = (x_{11}, x_{12}, x_{13}, \ldots),$$
$$\vec{x}_2 = (x_{21}, x_{22}, x_{23}, \ldots),$$
$$\vec{x}_3 = (x_{31}, x_{32}, x_{33}, \ldots), \ldots$$

is a Cauchy sequence in $\hat{X}$.

To show: There exists $\vec{z} = (z_1, z_2, \ldots)$ in $\hat{X}$ such that $\lim\limits_{n\to\infty} \vec{x}_n = \vec{z}$.

Using that $\overline{\varphi[X]}$ is $\hat{X}$,

for $k \in \mathbb{Z}_{>0}$ let $z_k \in X$ be such that $\hat{d}(\varphi(z_k), \vec{x}_k) < \frac{1}{k}$.

$\vec{x}_1 = (x_{11}, x_{12}, x_{13}, \ldots)$, $\quad \varphi(z_1) = (z_1, z_1, z_1, \ldots)$,

$\vec{x}_2 = (x_{21}, x_{22}, x_{23}, \ldots)$, $\quad \varphi(z_2) = (z_2, z_2, z_2, \ldots)$,

$\vec{x}_3 = (x_{31}, x_{22}, x_{33}, \ldots)$, $\quad \varphi(z_3) = (z_3, z_3, z_3, \ldots)$,

$\ldots \qquad\qquad \ldots$

To show: (ba) $\vec{z} = (z_1, z_2, \ldots)$ is a Cauchy sequence

(bb) $\lim_{n \to \infty} \vec{x}_n = \vec{z}$.

(ba) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>0}$ and $r > N$ and $s > N$ then $d(z_r, z_s) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>0}$ and $r > N$ and $s > N$ then $d(z_r, z_s) < \varepsilon$.

Let $N_1 = \left\lceil \frac{3}{\varepsilon} \right\rceil + 1$, so that $\frac{1}{N_1} < \frac{\varepsilon}{3}$.

Let $N_2 \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>0}$ and $r > N_2$ and $s > N_2$ then $\hat{d}(\vec{x}_r, \vec{x}_s) < \frac{\varepsilon}{3}$.

Let $N = \max(N_1, N_2)$.

To show: If $r, s \in \mathbb{Z}_{>0}$ and $r > N$ and $s > N$ then $d(z_r, z_s) < \varepsilon$.

Assume $r, s \in \mathbb{Z}_{>0}$ and $r > N$ and $s > N$.

To show: $d(z_r, z_s) < \varepsilon$.

$$
\begin{aligned}
d(z_r, z_s) &= \hat{d}(\varphi(z_r), \varphi(z_s)) \\
&\le \hat{d}(\varphi(z_r), \vec{x}_r) + \hat{d}(\vec{x}_r, \vec{x}_s) + \hat{d}(\vec{x}_s, \varphi(z_s)) \\
&< \frac{1}{r} + \frac{\varepsilon}{3} + \frac{1}{s} < \frac{1}{N_1} + \frac{\varepsilon}{3} + \frac{1}{N_1} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

So $\vec{z}$ is Cauchy.

(bb) To show: $\lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) = 0$.

$$
\begin{aligned}
\lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) &\le \lim_{n\to\infty} \left(\hat{d}(\vec{x}_n, \varphi(z_n)) + \hat{d}(\varphi(z_n, \vec{z})) \right) \\
&\le \lim_{n\to\infty} \left(\frac{1}{n} + \hat{d}(\varphi(z_n), \vec{z}) \right) \\
&= \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \hat{d}(\varphi(z_n), \vec{z}) \\
&= 0 + 0 = 0.
\end{aligned}
$$

So $(\hat{X}, \hat{d})$ is complete.

So $(\hat{X}, \hat{d})$ with φ is a completion. //