Completions

Let (X,d) be a metric space.

The completion of (X,d) is a metric space $(\hat{X},\hat{d})$ with an isometry $\varphi: X \to \hat{X}$ such that

$$(\hat{X},\hat{d}) \text{ is complete and } \overline{\varphi(X)} = \hat{X}.$$

HW (Uniqueness of completions). If $(\hat{X}_1,\hat{d}_1)$ with $\varphi_1: X \to \hat{X}_1$ and $(\hat{X}_2,\hat{d}_2)$ with $\varphi_2: X \to \hat{X}_2$ are completions of (X,d) then there exists $f: \hat{X}_1 \to \hat{X}_2$ such that

(a) f is an isometry,

(b) f is a bijection,

(c) $f \circ \varphi_1 = \varphi_2$.

$$\begin{tikzcd} & \hat{X}_1 \arrow[d, "f"] \\ X \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"'] & \hat{X}_2 \end{tikzcd}$$

Existence of completions

Let (X,d) be a metric space.

Let $\hat{X}$ be the set of Cauchy sequences $\vec{x}: \mathbb{Z}_{>0} \to X$

with $\vec{x} = \vec{y}$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$,

where $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ and $\vec{y}: \mathbb{Z}_{>0} \to X$, $n \mapsto y_n$.

Define $\hat{d}: \hat{X} \times \hat{X} \to \mathbb{R}_{\geq 0}$ by

$$d(\vec{x}, \vec{y}) = \lim_{n \to \infty} d(x_n, y_n),$$

where $\begin{array}{rl} \vec{x}: \mathbb{Z}_{\geq 0} & \to X \\ n & \mapsto x_n \end{array}$ and $\begin{array}{rl} \vec{y}: \mathbb{Z}_{\geq 0} & \to X \\ n & \mapsto y_n. \end{array}$

Define $\varphi: X \to \hat{X}$ by $\varphi(x) = (x, x, \ldots)$,

i.e. $\begin{array}{rl} \varphi(x): \mathbb{Z}_{\geq 0} & \to X \\ n & \mapsto x. \end{array}$

<u>Theorem</u> $(\hat{X}, \hat{d})$ with $\varphi: X \to \hat{X}$ is a completion of (X, d).

<u>Proof</u> To show: (a) $(\hat{X}, \hat{d})$ is a metric space

(b) $(\hat{X}, \hat{d})$ is complete

(c) $\varphi: X \to \hat{X}$ is an isometry

(d) $\overline{\varphi(X)} = \hat{X}$.

(c) To show: If $x, y \in X$ then $d(\varphi(x), \varphi(y)) = d(x, y)$.

Assume $x, y \in X$.

To show: $d(\varphi(x), \varphi(y)) = d(x, y)$

$$d(\varphi(x), \varphi(y)) = \lim_{n \to \infty} d(\varphi(x)_n, \varphi(y)_n)$$

$$= \lim_{n \to \infty} d(x, y) = d(x, y).$$

So φ is an isometry.

(a) To show: $(\hat{X}, \hat{d})$ is a metric space

To show: (aa) $\hat{d}: \hat{X} \times \hat{X} \to \mathbb{R}_{\geq 0}$ given by

$$\hat{d}(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n),$$ is a function.

(ab) If $\vec{x}, \vec{y} \in \hat{X}$ then $\hat{d}(\vec{x}, \vec{y}) = \hat{d}(\vec{y}, \vec{x})$

(ac) If $\vec{x} \in \hat{X}$ then $\hat{d}(\vec{x}, \vec{x}) = 0$.

(ad) If $\vec{x}, \vec{y} \in \hat{X}$ and $\hat{d}(\vec{x}, \vec{y}) = 0$ then $\vec{x} = \vec{y}$.

(ae) If $\vec{x}, \vec{y}, z \in \hat{X}$ then $\hat{d}(\vec{x}, \vec{y}) \leq \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y})$.

(aa) To show: If $\vec{x}, \vec{y} \in \hat{X}$ then there exists a unique $z \in \mathbb{R}_{\geq 0}$ such that $z = \lim_{n\to\infty} d(x_n, y_n)$.

Assume $\vec{x}, \vec{y} \in \hat{X}$ with $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$.
Let $d_1, d_2, \ldots$ be the sequence in $\mathbb{R}_{\geq 0}$ given by

$$d_n = d(x_n, y_n)$$

To show: There exists $z \in \mathbb{R}_{\geq 0}$ such that $z = \lim_{n\to\infty} d_n$

Since limits in metric spaces are unique when they exist, z will be unique if it exists, see Rubinstein Notes Proposition 2.9.

To show: $d_1, d_2, \ldots$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$
This will show that z exists since Cauchy sequences in $\mathbb{R}_{\geq 0}$ converge, since $\mathbb{R}_{\geq 0}$ is complete, see Rubinstein notes Theorem 4.6.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $|d_m - d_n| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $N = \max(N_1, N_2)$, where

N_1 is such that if $n, m > N_1$ then $d(x_m, x_n) < \frac{\varepsilon}{2}$,

N_2 is such that if $n, m > N_2$ then $d(y_m, y_n) < \frac{\varepsilon}{2}$.

(N_1 and N_2 exist since $\vec{x}$ and $\vec{y}$ are Cauchy sequences).

To show: If $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $|d_m - d_n| < \varepsilon$.

Assume $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$.

To show: $|d_m - d_n| < \varepsilon$.

$$|d_m - d_n| = |d(x_m, y_m) - d(x_n, y_n)|$$

$$< |d(x_n, x_m) + d(y_n, y_m)|,$$

since $d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_n, y_m)$.

~~and~~ So

$$|d_m - d_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $d_1, d_2, \ldots$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$.

So $z = \lim_{n \to \infty} d_n$ exists in $\mathbb{R}_{\geq 0}$.

(ab) To show: If $\vec{x}, \vec{y} \in \hat{X}$ then $\hat{d}(\vec{x}, \vec{y}) = \hat{d}(\vec{y}, \vec{x})$

Assume $\vec{x}, \vec{y} \in \hat{X}$ with $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$.

Since $d(x_n, y_n) = d(y_n, x_n)$,

$$\hat{d}(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \hat{d}(\vec{y}, \vec{x}).$$

(ac) To show: If $\vec{x} \in \hat{X}$ then $\hat{d}(\vec{x}, \vec{x}) = 0$.

Assume $\vec{x} \in \hat{X}$.

To show: $\hat{d}(\vec{x}, \vec{x}) = 0$.

Since $d(x_n, x_n) = 0$,

$$\hat{d}(\vec{x}, \vec{x}) = \lim_{n\to\infty} d(x_n, x_n) = \lim_{n\to\infty} 0 = 0.$$

(ad) To show: If $\vec{x}, \vec{y} \in \hat{X}$ and $\hat{d}(\vec{x}, \vec{y}) = 0$ then $\vec{x} = \vec{y}$.

Assume $\vec{x}, \vec{y} \in \hat{X}$ and $\hat{d}(\vec{x}, \vec{y}) = 0$.

To show: $\vec{x} = \vec{y}$.

This is a consequence of the definition of $\hat{X}$ which requires that $\vec{x} = \vec{y}$ if $\hat{d}(\hat{x}, \hat{y}) = 0$.

(ae) If $\vec{x}, \vec{y}, \vec{z} \in \hat{X}$ then $\hat{d}(\vec{x}, \vec{y}) \le \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y})$.

Assume $\vec{x}, \vec{y}, \vec{z} \in \hat{X}$.

To show: $\hat{d}(\vec{x}, \vec{y}) \le \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y})$.

$$\hat{d}(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} \left(d(x_n, z_n) + d(z_n, y_n)\right)$$