Convergence and boundedness

Let (X,d) be a metric space

Let $A \subseteq X$.

The set A is <u>bounded</u> if A satisfies:

there exists $M \in \mathbb{R}_{\geq 0}$ such that

if $a_1, a_2 \in A$ then $d(a_1, a_2) \leq M$.

<u>HW</u> Define the <u>diameter</u> of A,

the <u>distance between A and B</u>, and

the <u>distance between x and A</u>.

Convergence and boundedness

<u>HW</u>: Let (X,d) be a metric space and let $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$

be a sequence in X. Show that if $\vec{x}$ converges

then $\{x_1, x_2, x_3, \ldots\}$ is bounded.

<u>Proof</u> Assume $\vec{x}$ converges.

To show: $\{x_1, x_2, \ldots\}$ is bounded.

We know: There exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

To show: There exists $M \in \mathbb{R}_{\geq 0}$ such that

if $i,j \in \mathbb{Z}_{>0}$ then $d(x_i, x_j) \leq M$.

Let $N \in \mathbb{Z}_{>0}$ be such that if $n \in \mathbb{Z}_{>0}$ and $n \geq N$ then

$d(x_n, x) < 1$

Let $M = 2 \cdot \max\{1, d(x_1, x), \ldots, d(x_N, x)\}$

Completeness

Let (X,d) be a metric space.
A Cauchy sequence is a sequence $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$
such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $m,n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(x_m, x_n) < \varepsilon$.

A complete metric space is a metric space (X,d) such that

if $\vec{x}: \mathbb{Z}_{>0} \to X$ is a Cauchy sequence in X

then there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

HW (convergent implies Cauchy) Let (X,d) be a metric space and let $\vec{x}: \mathbb{Z}_{>0} \to X$ be a sequence in X. Show that

if there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$

then $\vec{x}: \mathbb{Z}_{>0} \to X$ is a Cauchy sequence in X.

Completeness and closure

HW Let (X,d) be a metric space and let $Y \subseteq X$.
Show that if X is complete and Y is closed then Y is complete.

HW Let (X,d) be a metric space and let $Y \subseteq X$.
Show that if Y is complete then Y is closed.

Examples of completeness

HW Show that $\mathbb{R}$ is complete.

HW Show that if $X_1, X_2, \ldots, X_m$ are complete then $X_1 \times X_2 \times \cdots \times X_m$ is complete.

HW Let X and Y be metric spaces and let

$$C_b(X,Y) = \{f : X \to Y \mid f \text{ is continuous and } f(X) \text{ is bounded}\}$$

with norm $\rho : C_b(X,Y) \times C_b(X,Y) \to \mathbb{R}_{\geq 0}$ given by

$$\rho(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

Show that

if Y is complete then $C_b(X,Y)$ is complete.

HW Let (X,d) be a metric space. Let

$$C_b(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and } f(X) \text{ is bounded}\}$$

with norm $\rho : C_b(X) \times C_b(X) \to \mathbb{R}_{\geq 0}$ given by

$$\rho(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

Show that $C_b(X)$ is complete.

Homeomorphisms and isometries

Let $(X, \mathcal{T})$ and $(Y, \mathcal{R})$ be topological spaces.
A homeomorphism from X to Y is a function $\varphi: X \to Y$ such that

(a) φ is continuous

(b) the inverse function $\varphi^{-1}: Y \to X$ exists (i.e. $\varphi: X \to Y$ is bijective)

(c) $\varphi^{-1}: Y \to X$ is continuous.

Let (X, d) and (Y, ρ) be metric spaces.
An isometry from X to Y is a function $\varphi: X \to Y$ such that

if $\cancel{\forall}$ $x_1, x_2 \in X$ then $\rho(\varphi(x_1), \varphi(x_2)) = d(x_1, x_2)$.

HW Show that if $\varphi: X \to Y$ is an isometry then φ is injective.

HW Show that $\varphi: \mathbb{Q} \to \mathbb{R}$ is an isometry that is not surjective.