Lecture 10: Metric and Hilbert Spaces 13 August 2014

Continuity and uniformly continuity

Let (X,d) and (C,ρ) be metric spaces.

Let $f: X \to C$ be a function.

The function $f: X \to C$ is continuous if f satisfies:

if $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in X$ and $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

The function $f: X \to C$ is uniformly continuous if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x \in X$ and $y \in X$ and $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

HW: Second definition The function $f: X \to C$ is continuous if and only if f satisfies

if $x \in X$ then $\lim_{y \to x} f(y) = f(x)$.

HW: Third definition The function $f: X \to C$ is continuous if and only if f satisfies

if $x \in X$ and $\vec{x}: \mathbb{Z}_{>0} \to X$, $n \mapsto x_n$ and $\lim_{n \to \infty} x_n = x$

then $\lim_{n \to \infty} f(x_n) = f(x)$.

HW: Fourth definition The function $f: X \to C$ is continuous if and only if f is continuous as a function between topological spaces i.e., if f satisfies

if V is open in C then $f^{-1}(V)$ is open in X.

Examples

HW Let $f: X \to Y$ and $g: Y \to Z$ be continuous. Show that $g \circ f$ is continuous.

HW Let $A \subseteq X$ and let $f: X \to Y$ be continuous. Show that $g: \begin{array}{l} A \to Y \\ a \mapsto f(a) \end{array}$ is continuous.

HW Let $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ be continuous. Show that $f: \begin{array}{l} X_1 \times X_2 \longrightarrow Y_1 \times Y_2 \\ (x_1, x_2) \longmapsto (f_1(x_1), f_2(x_2)) \end{array}$ is continuous.

HW Show that $\begin{array}{l} \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\ (x,y) \longmapsto x+y \end{array}$ is continuous.

HW Show that $\begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto -x \end{array}$ is continuous.

HW Show that $\begin{array}{l} \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\ (x,y) \longmapsto xy \end{array}$ is continuous

HW Show that $\begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto \frac{x}{1+x^2} \end{array}$ is uniformly continuous.

HW Show that if $f: X \to Y$ is uniformly continuous then $f: X \to Y$ is continuous

HW Show that $\begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto x^2 \end{array}$ is not uniformly continuous.

Sequences of functions

Examples (1) $f_1, f_2, \ldots$ defined by $f_n: [0,1) \to [0,1)$
$$x \mapsto x^n.$$

(2) $f_1, f_2, \ldots$ defined by $f_n: [0,1] \to [0,1]$
$$x \mapsto x^n$$

(3) $f_1, f_2, \ldots$ defined by $f_n: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
$$x \mapsto x^n.$$

Let (X,d) and (C,ρ) be metric spaces.
Let
$$F = \{\text{functions } f: X \to C\}$$
and define $d: F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by
$$d(f,g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}$$
(Warning $d: F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is not quite a metric).

Let $\vec{f}: \begin{array}{l} \mathbb{Z}_{>0} \to F \\ n \mapsto f_n \end{array}$ be a sequence of functions from X to C

and let $f: X \to C$ be a function.

The sequence $\vec{f}: \begin{array}{l} \mathbb{Z}_{>0} \to F \\ n \mapsto f_n \end{array}$ converges pointwise to f

if $\vec{f}$ satisfies

if $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(f_n(x), f(x)) < \varepsilon$.

The sequence $\vec{f}: \mathbb{Z}_{>0} \to F$, $n \mapsto f_n$ <u>converges</u> <u>uniformly</u> to f

if $\vec{f}$ satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $x \in X$ and $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(f_n(x), f(x)) < \varepsilon$.

<u>HW</u> <u>Second definition</u>

The sequence $\vec{f}: \mathbb{Z}_{>0} \to F$, $n \mapsto f_n$ <u>converges</u> <u>pointwise</u> to f

if $\vec{f}$ satisfies

if $x \in X$ then $\lim_{n \to \infty} d(f_n(x), f(x)) = 0$

The sequence $\vec{f}: \mathbb{Z}_{>0} \to F$, $n \mapsto f_n$ <u>converges</u> <u>uniformly</u> to f

if $\lim_{n \to \infty} d(f_n, f) = 0$.