(5) ~~B~~ Let $p \in \mathbb{R}_{>1}$ and let $q \in \mathbb{R}_{>1}$ be given by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(a) Define the normed vector space ℓ^p.

$$\ell^p = \{ (x_1, x_2, \dots) \mid x_i \in \mathbb{R} \text{ and } \|(x_1, x_2, \dots)\|_p < \infty \}$$

where

$$\|(x_1, x_2, \dots)\| = \left(\sum_{i \in \mathbb{Z}_{>0}} |x_i|^p \right)^{1/p}.$$

(b) If V and W are normed vector spaces let

$$B(V, W) = \left\{ T : V \to W \;\middle|\; \begin{array}{l} T \text{ is a linear operator} \\ \text{and } \|T\| \text{ is bounded} \end{array} \right\}$$

where

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} \;\middle|\; x \in V \right\}.$$

Recall the following from the Rubinstein notes:

<u>Theorem 11.8</u> If W is a Banach space then $B(V, W)$ is a Banach space

<u>Theorem 4.6</u> The space $\mathbb{R}$ with the usual metric is complete.

Together these imply that $B(V, \mathbb{R})$ is complete.
In part (c) we will show that

$$\ell^p = B(\ell^q, \mathbb{R}).$$

Thus ℓ^p is a Banach space.

(c) To show: ℓ^q is the dual of ℓ^p.

To show: $\ell^q = B(\ell^p, \mathbb{R})$.

Define

$$\begin{aligned} \varphi : \ell^q &\longrightarrow B(\ell^p, \mathbb{R}) \\ y &\longmapsto \varphi_y : \ell^p \longrightarrow \mathbb{R} \\ & \qquad\quad x \longmapsto \langle y, x\rangle \end{aligned}$$

where

$$\langle y, x\rangle = \sum_{i \in \mathbb{Z}_{>0}} y_i x_i, \quad \text{if } y = (y_1, y_2, \ldots) \text{ and } x = (x_1, x_2, \ldots).$$

To show: (ca) φ is a linear transformation.

(cb) φ is invertible

(cc) If $y \in \ell^q$ then $\|\varphi_y\| = \|y\|$.

(ca) To show: (caa) If $y_1, y_2 \in \ell^q$ then $\varphi(y_1+y_2) = \varphi(y_1) + \varphi(y_2)$

(cab) If $y \in \ell^q$ and $c \in \mathbb{R}$ then $\varphi(cy) = c\varphi(y)$.

(caa) Assume $y_1, y_2 \in \ell^q$.

To show: $\varphi(y_1+y_2) = \varphi(y_1) + \varphi(y_2)$

To show: If $x \in \ell^p$ then ~~$\varphi(y_1+y_2)$~~ $\varphi_{y_1+y_2}(x) = (\varphi_{y_1} + \varphi_{y_2})(x)$

Assume $x \in \ell^p$.

To show: $\varphi_{y_1+y_2}(x) = (\varphi_{y_1} + \varphi_{y_2})(x)$

$$\begin{aligned} \varphi_{y_1+y_2}(x) &= \langle y_1+y_2, x\rangle = \langle y_1, x\rangle + \langle y_2, x\rangle = \varphi_{y_1}(x) + \varphi_{y_2}(x) \\ &= (\varphi_{y_1} + \varphi_{y_2})(x). \end{aligned}$$

(cab) Assume $y \in \ell^q$ and $c \in \mathbb{R}$.

To show: $\varphi(cy) = c\varphi(y)$

To show: If $x \in \ell^p$ then $\varphi_{cy}(x) = (c\varphi_y)(x)$.

Assume $x \in \ell^p$.

To show: $\varphi_{cy}(x) = (c\varphi_y)(x)$

$$\varphi_{cy}(x) = \langle cy, x \rangle = c\langle y, x \rangle = c(\varphi_y(x)) = (c\varphi_y)(x).$$

So ~~this~~ $\varphi: \ell^q \to B(\ell^p, \mathbb{R})$ is a linear transformation.

(cb) To show: $\varphi: \ell^2 \to B(\ell^p, \mathbb{R})$ is invertible.

To show: There exists $\psi: B(\ell^p, \mathbb{R}) \to \ell^q$ such that $\varphi \circ \psi = id$ and $\psi \circ \varphi = id$.

Let $\psi: B(\ell^p, \mathbb{R}) \to \ell^q$ be given by

$$\psi(\gamma) = (\gamma(e_1), \gamma(e_2), \ldots)$$

where $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 in the i^{th} spot.

To show: (cba) $\varphi \circ \psi = id$

(cbb) $\psi \circ \varphi = id$.

(cba) To show: If $\gamma \in B(\ell^p, \mathbb{R})$ then $\varphi(\psi(\gamma)) = \gamma$.

Assume $\gamma \in B(\ell^p, \mathbb{R})$

To show: $\varphi(\psi(\gamma)) = \gamma$.

To show: If $x \in \ell^p$ then $\varphi(\psi(\gamma))(x) = \gamma(x)$.

Assume $x \in \ell^p$. Let $x = (x_1, x_2, \ldots)$.

To show: $\varphi(\Psi(\gamma))(x) = \gamma(x)$.

$$\begin{aligned}\varphi(\Psi(\gamma))(x) &= \varphi(\gamma(e_1), \gamma(e_2), \ldots)(x) \\ &= \langle (\gamma(e_1), \gamma(e_2), \ldots), (x_1, x_2, \ldots) \rangle \\ &= \sum_{i \in \mathbb{Z}_{>0}} \gamma(e_i) x_i = \gamma\Big(\sum_{i \in \mathbb{Z}_{>0}} x_i e_i \Big) = \gamma(x).\end{aligned}$$

(cbb) To show: $\Psi \circ \varphi = id$.

To show: If $y \in \ell^q$ then $\Psi(\varphi(y)) = y$.

Assume $y \in \ell^q$. Let $y = (y_1, y_2, \ldots)$.

To show: $\Psi(\varphi(y)) = y$.

$$\Psi(\varphi(y)) = \Psi(\varphi_y) = (\varphi_y(e_1), \varphi_y(e_2), \ldots) = (y_1, y_2, \ldots),$$

since $\varphi_y(e_i) = \langle y, e_i \rangle = \langle (y_1, y_2, \ldots), (0, \ldots, 0, 1, 0, \ldots) \rangle = y_i$.

So $\Psi(\varphi(y)) = y$.

(cc) To show: If $y \in \ell^q$ then $\|\varphi_y\| = \|y\|_q$.

Assume $y \in \ell^q$. Let $y = (y_1, y_2, \ldots)$.

To show: (cca) $\|\varphi_y\| \le \|y\|_q$

(ccb) $\|\varphi_y\| \ge \|y\|_q$.

(cca) To show: If $x \in \ell^p$ then $|\varphi_y(x)| \le \|x\|_p \|y\|_q$.

Assume $x \in \ell^p$. Let $x = (x_1, x_2, \ldots)$.

Then
$$|\varphi_y(x)| = \left|\sum_{n\in\mathbb{Z}_{>0}} x_n y_n\right| \le \|x\|_p \, \|y\|_q$$

by Hölder's inequality. So $\|\varphi_y\| \le \|y\|_q$

(ccb) To show: $\|\varphi_y\| \ge \|y\|_q$.

To show: There exists $x \in \ell^p$ with $|\varphi_y(x)| \ge \|x\|_p \, \|y\|_q$.

Let
$$x = (\mathrm{sgn}(y_1)|y_1|^{q-1}, \mathrm{sgn}(y_2)|y_2|^{q-1}, \ldots)$$

Then
$$\|x\|_p = \left(\sum_{n\in\mathbb{Z}_{>0}} |x_n|^p\right)^{1/p} = \left(\sum_{n\in\mathbb{Z}_{>0}} |\mathrm{sgn}(y_n)|y_n|^{q-1}|^p\right)^{1/p}$$

$$= \left(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^{pq-p}\right)^{1/p} = \left(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^{pq(1-\frac{1}{q})}\right)^{1/p}$$

$$= \left(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^{pq\frac{1}{p}}\right)^{\frac{1}{p}} = \left(\left(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^q\right)^{1/q}\right)^{q\frac{1}{p}}$$

$$= \|y\|_q^{q\frac{1}{p}} = \|y\|_q^{q(1-\frac{1}{q})} = \|y\|_q^{q-1}.$$

So
$$|\varphi_y(x)| = \left|\sum_{n\in\mathbb{Z}_{>0}} x_n y_n\right| = \left|\sum_{n\in\mathbb{Z}_{>0}} (\mathrm{sgn}(y_n)|y_n|)(\mathrm{sgn}(y_n)|y_n|^{q-1})\right|$$

$$= \sum_{n\in\mathbb{Z}_{>0}} |y_n|^q = \|y\|_q^q = \|y\|_q \, \|y\|_q^{q-1} = \|y\|_q \, \|x\|_p.$$

So
$$\|\varphi_y\| \ge \|y\|_q.$$

Let $D : C'[0,1] \to C[0,1]$ be given by $Df = \frac{df}{dt}$.

Let $C[0,1]$ have norm given by $\|f\| = \sup\{|f(t)| \mid t \in [0,1]\}$

Let $C'[0,1]$ have norm given by $\|f\|_0 = \|f\| + \|\frac{df}{dt}\|$.

(a) Show that D is a bounded linear operator

with $\|D\| = 1$.

To show: $\|D\| = 1$.

To show: (aa) $\|D\| \geq 1$

(ab) $\|D\| \leq 1$.

(aa) ~~Since~~ If $n \in \mathbb{Z}_{>0}$ then

$$\frac{\|Dt^n\|}{\|t^n\|_0} = \frac{\|nt^{n-1}\|}{\|nt^{n-1}\| + \|t^n\|} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}.$$

So $\|D\| \geq \frac{1}{1+\frac{1}{n}}$ for $n \in \mathbb{Z}_{>0}$.

So $\|D\| \geq 1$.

(ab) ~~Since~~ Let $f \in C'[0,1]$. Since

$$\|Df\| = \|\tfrac{df}{dt}\| \leq \|f\| + \|\tfrac{df}{dt}\| = \|f\|_0$$

then $\frac{\|Df\|}{\|f\|_0} \leq 1$.

So $\|D\| \leq 1$.

(b) Show that $D : (C^1[0,1], \|\ \|) \to (C[0,1], \|\ \|)$ is unbounded.

a

To show: If $n \in \mathbb{Z}_{\geq 0}$ then $\|D\| \geq n$.

Assume $n \in \mathbb{Z}_{\geq 0}$.

Since $\dfrac{\|Dt^n\|}{\|t^n\|} = \dfrac{\|n t^{n-1}\|}{\|t^n\|} = \dfrac{n}{1} = n$, then

$\|D\| \geq n$.

So D is unbounded.

(7) Let $(a_1, a_2, \ldots)$ be a bounded sequence in $\mathbb{C}$.
Define $T : l^2 \to l^2$ by

$$T(b_1, b_2, \ldots) = T(0, a_1 b_1, a_2 b_2, \ldots).$$

(a) Show that T is a bounded linear operator and find $\|T\|$.

Let $e_i = (0, 0, \ldots, \overset{ith}{0, 1}, 0, \ldots)$ so that $\|e_i\| = 1$.
Then

$$\|Te_i\|^2 = |a_i|^2 = |a_i|^2 \|e_i\|^2, \quad \text{so } \|T\| \geq |a_i|.$$

So $\|T\| \geq \sup\{|a_1|, |a_2|, \ldots\}$.

To show: $\|T\| = \sup\{|a_1|, |a_2|, \ldots\}$.

To show: $\|T\| \leq \sup\{|a_1|, |a_2|, \ldots\}$.

Let $b = (b_1, b_2, \ldots) \in l^2$. Then

$$\|Tb\|^2 = \sum_{i \in \mathbb{Z}_{>0}} |a_i b_i|^2 \leq \sum_{i \in \mathbb{Z}_{>0}} |a_i|^2 |b_i|^2$$

$$\leq \sum_{i \in \mathbb{Z}_{>0}} k^2 |b_i|^2 = k^2 \|b\|^2, \text{ where } k = \sup\{|a_1|, |a_2|, \ldots\}.$$

So $\|T\| \leq k = \sup\{|a_1|, |a_2|, \ldots\}$.

So $\|T\| = \sup\{|a_1|, |a_2|, \ldots\}$.

(b) Compute T^*.

Since ℓ^2 is a Hilbert space, T^* is determined by

$$\langle b, T^*c\rangle = \langle Tb, c\rangle = \sum_{i\in\mathbb{Z}_{>0}} a_i b_i \overline{c_{i+1}} = \sum_{i\in\mathbb{Z}_{>0}} b_i \overline{(\overline{a_i} c_{i+1})}.$$

So $T^*c = (\overline{a_1}c_2, \overline{a_2}c_3, \overline{a_3}c_4, \ldots)$.

(c) Show that $T^*T \neq TT^*$ whenever $T \neq 0$.

To show: If $T^*T = TT^*$ then $T=0$.

Assume $T^*T = TT^*$.

Let $c = (c_1, c_2, \ldots) \in \ell^2$. Then

$$T^*T(c) = T^*(0, a_1c_1, a_2c_2, \ldots)$$
$$= (\overline{a_1}a_1c_1, \overline{a_2}a_2c_2, \ldots) = (|a_1|^2c_1, |a_2|^2c_2, \ldots)$$

and

$$TT^*(c) = T(\overline{a_1}c_2, \overline{a_2}c_3, \ldots) = (0, a_1\overline{a_1}c_2, a_2\overline{a_2}c_3, \ldots)$$
$$= (0, |a_1|^2c_2, |a_2|^2c_3, \ldots)$$

Thus $T^*T = TT^*$ implies

$$|a_1|^2c_1 = 0,\quad |a_2|^2c_2 = |a_1|^2c_2,\quad |a_3|^3c_3 = |a_2|^2c_3, \ldots$$

for all $c = (c_1, c_2, \ldots) \in \ell^2$.

So $T^*T = TT^*$ implies $|a_1|^2 = 0$, $|a_2|^2 = |a_1|^2$, $|a_3|^2 = |a_2|^2, \ldots$

So $T^*T = TT^*$ implies $0 = a_1 = a_2 = \ldots$.

So $T = 0$.

(d) Find the eigenvalues of T^*.

Assume $c = (c_1, c_2, \ldots) \in \ell^2$ is an eigenvector of T^* with eigenvalue λ.

Then

$$\lambda(c_1, c_2, \ldots) = T^* c = (\bar{a}_1 c_2, \bar{a}_2 c_3, \ldots) \quad \text{so that}$$

$\bar{a}_1 c_2 = \lambda c_1$, $\bar{a}_2 c_3 = \lambda c_2, \ldots$ and

$$c_2 = \frac{\lambda}{\bar{a}_1} c_1, \quad c_3 = \frac{\lambda}{\bar{a}_2} c_2 = \frac{\lambda}{\bar{a}_1 \bar{a}_2} c_1, \ldots$$

So $c = \left(c_1, \frac{\lambda}{\bar{a}_1} c_1, \frac{\lambda^2}{\bar{a}_1 \bar{a}_2} c_1, \ldots\right) = c_1 \left(1, \frac{\lambda}{\bar{a}_1}, \frac{\lambda^2}{\bar{a}_1 \bar{a}_2}, \ldots\right)$

Since $c \in \ell^2$,

$$1 + \left|\frac{1}{\bar{a}_1}\right| |\lambda| + \left|\frac{1}{\bar{a}_1 \bar{a}_2}\right| |\lambda|^2 + \cdots \quad \text{converges.}$$

By the root test, this series converges if

$$\lim_{n \to \infty} \left(\frac{|\lambda|^n}{|a_1 a_2 \cdots a_n|}\right)^{1/n} = \limsup_{n \in \mathbb{Z}_{>0}} \left(\frac{|\lambda|^n}{|a_1 a_2 \cdots a_n|}\right)^n < 1.$$

So the series converges if

$$|\lambda| < \limsup_{n \in \mathbb{Z}_{>0}} |a_1 a_2 \cdots a_n|^{1/n}.$$

and the series diverges if

$$|\lambda| > \limsup_{n \in \mathbb{Z}_{>0}} |a_1 a_2 \cdots a_n|^{1/n}.$$

So $\lambda \in \mathbb{C}$ with $|\lambda| < L$ are eigenvalues of T^*, where $L = \limsup_{n \in \mathbb{Z}_{>0}} |a_1 a_2 \cdots a_n|^{1/n}$.

(8) Let $(a_{ij}) \in M_{\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}}(\mathbb{C})$ be such that

(1) if $j \in \mathbb{Z}_{>0}$ then $c_j = \sum_i |a_{ij}|$ converges,

(2) $(c_1, c_2, \ldots)$ is bounded.

Let $c = \sup\{c_1, c_2, \ldots\}$. Let $T: \ell^1 \to \ell^1$ be given by

$$T(b_1, b_2, \ldots) = \left(\sum_j a_{1j} b_j, \sum_j a_{2j} b_j, \ldots\right)$$

(in matrix notation

$$\begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_j a_{1j} b_j \\ \sum_j a_{2j} b_j \\ \vdots \end{pmatrix}.$$

)

Show that T is a bounded linear operator and $\|T\| = c$.

To show: (a) T is a linear operator

(b) $T: \ell^1 \to \ell^1$ is well defined

(c) $\|T\| \le c$

(d) $\|T\| \ge c$.

(a) To show: (aa) If $b = (b_1, b_2, \ldots)$ and $b' = (b_1', b_2', \ldots)$ then

$$T(b + b') = T(b) + T(b')$$

(ab) If $b = (b_1, b_2, \ldots)$ and $\lambda \in \mathbb{C}$ then

$$T(\lambda b) = \lambda T(b).$$

(aa) Let $b = (b_1, b_2, \ldots)$ and $b' = (b'_1, b'_2, \ldots)$.

To show: $T(b+b') = T(b) + T(b')$.

To show: $T(b+b')_i = (T(b)+T(b'))_i$.

$$T(b+b')_i = T((b_1+b'_1, b_2+b'_2, \ldots))_i$$
$$= \sum_j a_{ij}(b_j + b'_j) = \sum_j a_{ij} b_j + \sum_j a_{ij} b'_j$$

and $(T(b)+T(b'))_i = \sum_j a_{ij} b_j + \sum_j a_{ij} b'_j$.

So $T(b+b') = T(b) + T(b')$.

(ab) Let $b = (b_1, b_2, \ldots)$ and $\lambda \in \mathbb{C}$.

To show: $T(\lambda b) = \lambda T(b)$

To show: $T(\lambda b)_i = (\lambda T(b))_i$

$$T(\lambda b)_i = \sum_j a_{ij} (\lambda b)_j = \sum_j a_{ij} (\lambda b_j)$$
$$= \lambda \sum_j a_{ij} b_j = \lambda T(b)_i = (\lambda T(b))_i$$

So $T(\lambda b) = \lambda T(b)$.

So T is a linear operator if it is a function.

(b) To show: $T: \ell' \to \ell'$ is well defined.

To show (ba) If $b = (b_1, b_2, \ldots) \in \ell'$ then Tb is defined.

(bb) If $b = (b_1, b_2, \ldots) \in \ell'$ then $Tb \in \ell'$.

(ba) Let $b=(b_1,b_2,\ldots)\in \ell^1$.

To show: Tb is defined.

To show: If $i\in\mathbb{Z}_{>0}$ then $(Tb)_i$ is defined.

Assume $i\in\mathbb{Z}_{>0}$. Then

$(Tb)_i = \sum_j a_{ij}b_j$ which converges

since $\sum_j |a_{ij}|$ converges and $\sum_j |b_j|$ converges.

(More details: By the Cauchy-Schwarz inequality

$$\sum_j |a_{ij}||b_j| = \langle (|a_{i1}|,|a_{i2}|,\ldots),(|b_1|,|b_2|,\ldots)\rangle$$
$$\le \Big(\sum_j |a_{ij}|\Big)\Big(\sum_j |b_j|\Big) = \|(a_{ij})_{j\in\mathbb{Z}_{>0}}\|\,\|(b_j)_{j\in\mathbb{Z}_{>0}}\|$$

and since $\sum_j |a_{ij}||b_j|$ converges, $\sum_j a_{ij}b_j$ converges.)

(bb) ~~To~~ Show: If $b=(b_1,b_2,\ldots)\in\ell^1$ then $Tb\in\ell^1$.

Assume $(b_1,b_2,\ldots)\in\ell^1$

To show: $Tb\in\ell^1$

To show: $\sum_i |T(b)_i|$ converges.

$$\sum_i |T(b)_i| = \sum_i \Big|\sum_j a_{ij}b_j\Big| \le \sum_i\sum_j |a_{ij}b_j| = \sum_j\sum_i |a_{ij}||b_j|$$
$$\le \sum_j\sum_i a_{ij}|b_j| \le \sum_j c|b_j| = c\|b\| \quad (*)$$

So $Tb\in\ell^1$.

(c) To show: $\|T\| = c$.

Since $Te_j = (a_{1j}, a_{2j}, \ldots)$ then

$$\|Te_j\| = \sum_{i \in \mathbb{Z}_{>0}} |a_{ij}| = c_j = c_j\|e_j\|$$

So $\|T\| \geq c_j$.

So $\|T\| \geq c$, since $c = \sup\{|c_1|, |c_2|, \ldots\}$.

To show: $\|T\| \leq c$.

Let $b = (b_1, b_2, \ldots) \in \ell^1$. By the mysterious computation in $(*)$,

$$\|Tb\| = \sum_{i \in \mathbb{Z}_{>0}} |T(b)_i| \leq c\|b\|.$$

So $\|T\| \leq c$.

So $\|T\| = c$.