(5) Determine whether the following sequences of functions converge uniformly.

(a) $f_n(x) = e^{-nx^2}$, $x \in [0,1]$.

(b) $g_n(x) = e^{-x^2/n}$, $x \in [0,1]$.

(c) $g_n(x) = e^{-x^2/n}$, $x \in \mathbb{R}$.

Let (X, ρ) be a metric space and let $f_n : X \to \mathbb{R}$, $n \in \mathbb{Z}_{>0}$, be a sequence of functions from X to $\mathbb{R}$. Assume that $f : X \to \mathbb{R}$ defined by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \text{is well defined.}$$

The sequence $f_1, f_2, f_3, \ldots$ converges uniformly to f if

$$\lim_{n\to\infty} \left(\sup\{ \rho(f_n(x), f(x)) \mid x \in X \} \right) = 0.$$

(a) Define $f_n : [0,1] \to \mathbb{R}$ by

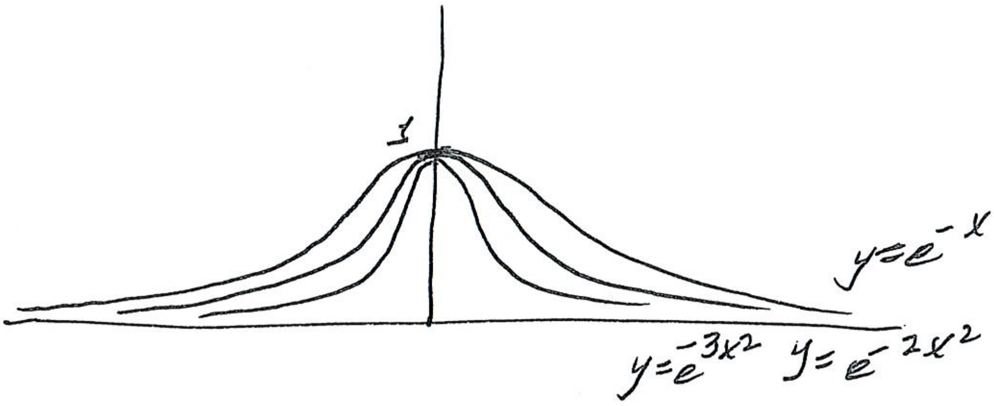
$$f_n(x) = e^{-nx^2} = e^{-(\sqrt{n}x)^2}, \quad \text{for } n \in \mathbb{Z}_{>0}.$$

Then

$$\lim_{n\to\infty} e^{-n\cdot 0^2} = e^{-0} = 1 \quad \text{and} \quad \lim_{n\to\infty} e^{-n\cdot 1^2} = e^{-\infty} = 0$$

and if $x \in (0,1)$ then

$$\lim_{n\to\infty} e^{-nx^2} = e^{-\infty} = 0.$$



Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \in (0,1]. \end{cases}$$

Let $n \in \mathbb{Z}_{>0}$. Then

$$\sup\{\rho(f_n(x), f(x)) \mid x \in X\}$$
$$= \sup\left(\{ |e^{-nx^2} - 0| \mid x \in (0,1]\} \cup \{|1-1|\}\right)$$
$$= \sup\{ e^{-nx^2} \mid x \in (0,1]\} = 1.$$

So

$$\lim_{n\to\infty} \left(\sup\{ |e^{-nx^2}| \mid x \in (0,1]\}\right) = \lim_{n\to\infty} 1 = 1.$$

So $f_1, f_2, f_3, \ldots$ is not uniformly convergent.

(b) Define $g_n : [0,1] \to \mathbb{R}$ by

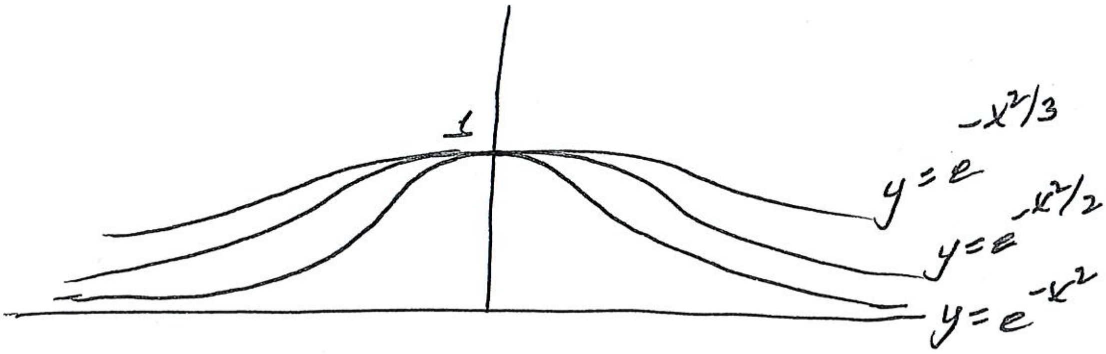
$$g_n(x) = e^{-x^2/n} = e^{-(x/\sqrt{n})^2}, \quad \text{for } n \in \mathbb{Z}_{>0}.$$

Then

$$\lim_{n\to\infty} e^{-0^2/n} = e^{-0} = 1 \text{ and } \lim_{n\to\infty} e^{-1^2/n} = e^{-1/\infty} = e^{-0} = 1$$

and if $x \in (0,1)$ then

$$\lim_{n\to\infty} e^{-x^2/n} = e^{-0} = 1.$$



Let $g : [0,1] \to \mathbb{R}$ be given by $g(x) = 1$.

Let $n \in \mathbb{Z}_{>0}$. Then

$$\sup\{\rho(g_n(x), g(x)) \mid x \in X\}$$
$$= \sup\{|e^{-x^2/n} - 1| \mid x \in [0,1]\}$$
$$= |e^{-1^2/n} - 1| = 1 - e^{-1/n}.$$

So $\lim_{n\to\infty}(\sup\{|e^{-x^2/n} - 1| \mid x \in [0,1]\}) = \lim_{n\to\infty} 1 - e^{-1/n}$

$$= 1 - e^{-1/\infty} = 1 - e^0 = 1 - 1 = 0.$$

So $g_1, g_2, g_3, \ldots$ uniformly converges to g.

(c) Define $g_n : \mathbb{R} \to \mathbb{R}$ by

$$g_n(x) = e^{-x^2/n} = e^{-(x/\sqrt{n})^2}, \quad \text{for } n \in \mathbb{Z}_{>0}.$$

Let $x \in \mathbb{R}$. Then

$$\lim_{n\to\infty} e^{-x^2/n} = e^{-x^2/\infty} = e^{-0} = 1.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 1$.
Let $n \in \mathbb{Z}_{>0}$. Then

$$\sup\{\rho(g_n(x), g(x)) \mid x \in X\}$$

$$= \sup\{|e^{-x^2/n} - 1| \mid x \in \mathbb{R}\}$$

$$= |e^{-\infty^2/n} - 1| = |e^{-\infty} - 1| = |0 - 1| = 1.$$

So

$$\lim_{n\to\infty} \sup\{\rho(g_n(x), g(x)) \mid x \in X\} = \lim_{n\to\infty} 1 = 1.$$

So $g_1, g_2, \ldots$ is not uniformly convergent.

(6) Let $X = \left\{ (x_1, x_2, \ldots) \;\middle|\; \begin{array}{l} x_i \in \mathbb{R} \text{ and all but a} \\ \text{finite number of } x_i \text{ are zero} \end{array} \right\}$

and define $d: X \times X \longrightarrow \mathbb{R}_{\geq 0}$ by

$$d(x,y) = \sup\{ |x_i - y_i| \mid i \in \mathbb{Z}_{>0} \}.$$

Let $f_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots, 0)$.

a) Show that $\{f_n\}$ is a Cauchy sequence in X.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(f_m, f_n) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(f_m, f_n) < \varepsilon$.

Let $N = \frac{1}{\varepsilon}$.

Assume $m, n \in \mathbb{Z}_{>0}$, $m > N$ and $n > N$ and $m < n$.

To show: $d(f_n, f_m) < \varepsilon$.

$$d(f_n, f_m) = \sup\{ 1-1, \tfrac{1}{2}-\tfrac{1}{2}, \ldots, \tfrac{1}{m}-\tfrac{1}{m}, \tfrac{1}{m+1}, \tfrac{1}{m+2}, \ldots, \tfrac{1}{n}, 0-0, 0-0, \ldots \}$$

$$= \frac{1}{m+1} < \frac{1}{N+1} = \frac{1}{\frac{1}{\varepsilon}+1} = \frac{\varepsilon}{\varepsilon+1} < \frac{\varepsilon}{1} = \varepsilon.$$

So $\{f_n\}$ is a Cauchy sequence in X.

The limit of $\{f_1, f_2, \ldots\}$ is the sequence

$$f = (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots)$$

If $k \in \mathbb{Z}_{>0}$ the k^{th} entry of f is $\frac{1}{k}$ which is not equal to 0. So all entries of f are nonzero.

So $f \notin X$.

So $\{f_1, f_2, \ldots\}$ is a Cauchy sequence in X which does not converge to a point in X.

So X is not complete.

(7) Let X be a set, $X \neq \phi$, and let (Y,d) be a complete metric space. Let $f: X \to Y$ be an injective function. Define $d_f: X \times X \to \mathbb{R}_{\geq 0}$ by

$$d_f(x,y) = d(f(x), d(f(y)).$$

(a) Show that d_f is a metric on X.

To show: (a) If $x, y \in X$ then $d_f(x,y) = d_f(y,x)$.
(b) If $x \in X$ then $d_f(x,x) = 0$.
(c) If $x, y \in X$ and $d_f(x,y) = 0$ then $x = y$
(d) If $x, y, z \in X$ then $d_f(x,y) \leq d_f(x,z) + d_f(z,y)$.

(a) Assume $x, y \in X$.
To show: $d_f(x,y) = d_f(y,x)$.

$d_f(x,y) = d(f(x), f(y)) = d(f(y), d(x)) = d_f(y,x)$.

(b) Assume $x \in X$.
To show: $d_f(x,x) = 0$.

$d_f(x,x) = d(f(x), f(x)) = 0$.

(c) Assume $x, y \in X$ and $d_f(x,y) = 0$.
To show: $x = y$.

Since $0 = d_f(x,y) = d(f(x), f(y))$ and d is a metric, then $f(x) = f(y)$.
Since $f: X \to Y$ is injective and $f(x) = f(y)$ then $x = y$.

(d) Assume $x, y, z \in X$.

To show: $d_f(x,y) \le d_f(x,z) + d_f(z,y)$

$$\begin{aligned} d_f(x,y) &= d(f(x), f(y)) \\ &\le d(f(x), f(z)) + d(f(z), f(y)) \\ &= d_f(x,z) + d_f(z,y) \end{aligned}$$

(7b) Show that (X, d_f) is a complete metric space if $f(X)$ is a closed subset of Y.

Proof Assume $f(X)$ is a closed subset of Y.

Since $f: X \to Y$ is injective and $f: X \to f(X)$ is surjective, then $f: X \to f(X)$ is bijective.

Since $d_f(x,y) = d(f(x), f(y))$ for $x, y \in X$ then

$$f: (X, d_f) \to (f(X), d) \text{ is an isometry.}$$

To show: (X, d_f) is complete.

To show: $(f(X), d)$ is complete.

To show: If $z_1, z_2, \ldots$ is a Cauchy sequence in $f(X)$ then $z_1, z_2, \ldots$ converges with $\lim_{n\to\infty} z_n$ in $f(X)$.

Assume $z_1, z_2, \ldots$ is a Cauchy sequence in $f(X)$.

Then $z_1, z_2, \ldots$ is a Cauchy sequence in Y.

Since Y is complete $z = \lim_{n\to\infty} z_n$ exists with $z \in Y$.

Since z is a close point of $z_1, z_2, \dots$, then
z is a close point of $f(X)$.

So $z \in f(X)$

So $f(X)$ is complete.

So (X, d_f) is complete.

(8) Let $f(x) = \frac{2}{2+x}$ for $x \in \mathbb{R}_{\geq 0}$.

(a) Show that f defines a contraction mapping $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Solution:

A function $f: X \to X$ is a contraction mapping if there exists $\alpha \in (0,1)$ such that

if $x, y \in X$ ~~then~~ $d(f(x), f(y)) \leq \alpha d(x,y)$.

Let $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given by $f(x) = \frac{2}{2+x}$.

Let $\alpha = \frac{1}{2}$.

To show: If $x, y \in \mathbb{R}_{\geq 0}$ then $d(f(x), f(y)) \leq \alpha d(x,y)$.

Assume $x, y \in \mathbb{R}_{\geq 0}$.

To show: $|f(y) - f(x)| \leq \alpha |y - x|$.

$$|f(y) - f(x)| = \left| \frac{2}{2+x} - \frac{2}{2+y} \right| = 2 \left| \frac{2+y-(2+x)}{(2+x)(2+y)} \right|$$

$$= \frac{2}{(2+x)(2+y)} |y-x| \leq \frac{2}{4} |y-x| = \frac{1}{2} |y-x|$$

So f is a contraction mapping.

The Banach fixed point theorem gives that the sequence

$$x_{n+1} = f(x_n) \quad \text{for} \quad x_0 \in X$$

converges to the (unique) fixed point of f.

In our case the fixed point is $p \in \mathbb{R}_{>0}$ such that

$$p = f(p) = \frac{2}{2+p}.$$

So $p^2 + 2p = 2$ and $p = \dfrac{-2 \pm \sqrt{4+4\cdot 2}}{2} = -1 \pm \sqrt{3}$.

Since $p \in \mathbb{R}_{>0}$ then $p = -1 + \sqrt{3}$.

(9) Let X be a connected topological space.

Let $f: X \to \mathbb{R}$ be continuous with $f(X) \subseteq \mathbb{Q}$.

Show that f is a constant function.

Solution:

To show: If f is not a constant function then X is not connected.

Assume f is not a constant function.

Let $a, b \in f(X)$ with $a < b$.

Let $z \in \mathbb{R}$, $z \notin \mathbb{Q}$ with $a < z < b$.

Let

$$A = f^{-1}((-\infty, z)) \quad \text{and} \quad B = f^{-1}((z, \infty))$$

Then

$A \neq \phi$ since $a \in A$,

$B \neq \phi$ since $b \in B$,

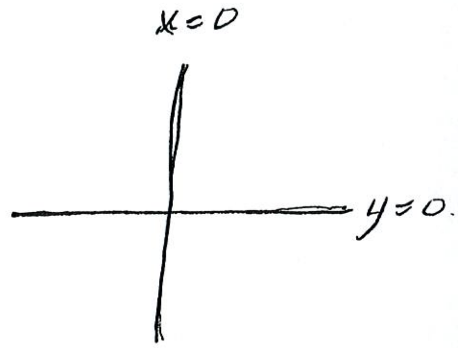
$A \cap B = \phi$ since $(-\infty, z) \cap (z, \infty) = \phi$.

Since $z \in \mathbb{Q}$ then $z \notin f(X)$ and since $(-\infty, z) \cup (z, \infty) = \mathbb{R} - \{z\}$ then $A \cup B = X$.

So X is not connected. //.

(10) Show that $X = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ is not homeomorphic to $\mathbb{R}$.

Solution



Let $X = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$

$= \{(x,y) \in \mathbb{R}^2 \mid x = 0\}$

$\cup \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$.

Assume that $f : X \to \mathbb{R}$ is a homeomorphism.

Let $a = f((0,0))$.

Then $f : X - \{(0,0)\} \to \mathbb{R} - \{a\}$ is a homeomorphism

So $X - \{(0,0)\}$ and $\mathbb{R} - \{a\}$ have the same number of connected components.

Since $X - \{(0,0)\}$ has 4 connected components and $\mathbb{R} - \{a\}$ has 2 connected components, this is a contradiction.

So X is not homeomorphic to $\mathbb{R}$.