§5.1 Definition of Hilbert space

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.

Let $\bar{\ }: \mathbb{C} \to \mathbb{C}$ denote complex conjugation.

Let H be a vector space over $\mathbb{K}$.

An <u>inner product</u> on H is a function

$$\begin{aligned} H \times H &\longrightarrow \mathbb{K} \\ (x,y) &\longmapsto \langle x,y \rangle \end{aligned} \quad \text{such that}$$

(a) if $x,y,z \in H$ then $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$,

(b) if $\alpha \in \mathbb{K}$ and $x,y \in H$ then $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$,

(c) if $x,y \in H$ then $\langle x,y\rangle = \overline{\langle y,x\rangle}$,

(d) if $x \in H$ then $\langle x,x\rangle \in \mathbb{R}_{\geq 0}$,

(e) if $x \in H$ and $\langle x,x\rangle = 0$ then $x = 0$.

Theorem 5.1 Let H be a vector space over $\mathbb{K}$.
Let $\langle \cdot,\cdot \rangle : H \times H \to \mathbb{K}$ be an inner product on H. Define

$$\|\cdot\| : H \longrightarrow \mathbb{R}_{\geq 0} \quad \text{by} \quad \|x\| = \sqrt{\langle x,x\rangle}.$$

(a) If $x,y \in H$ then $|\langle x,y\rangle| \leq \|x\| \cdot \|y\|$.

(b) If $x,y \in H$ then $\|x+y\| \leq \|x\| + \|y\|$.

(c) H with $\|\cdot\|$ is a normed vector space.

(d) If $x,y \in X$ and $\langle x,y\rangle = 0$ then

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2.$$

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.

A <u>Hilbert space</u> is a vector space over $\mathbb{K}$ with an inner product $\langle\,,\,\rangle : H \times H \to \mathbb{K}$ such that, with respect to $\|\cdot\| : H \to \mathbb{R}_{\geq 0}$ given by $\|x\| = \sqrt{\langle x, x\rangle}$,

H is a complete normed vector space.

<u>Examples</u>

(1) $H = \mathbb{R}^n$ with inner product $\langle\,,\,\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n, \quad \text{for } x = (x_1, \ldots, x_n), \; y = (y_1, \ldots, y_n).$$

(2) $H = \ell^2 = \{ x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{C} \text{ and } \|x\| < \infty \}$ with inner product $\langle\,,\,\rangle : \ell^2 \times \ell^2 \to \mathbb{C}$ given by

$$\langle x, y\rangle = \sum_{k \in \mathbb{Z}_{>0}} x_k \overline{y_k},$$

for $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$

(3) Let $\Omega \subseteq \mathbb{R}^n$ be open. Let

$$H = L^2(\Omega; \mathbb{R}) = \{ f : \Omega \to \mathbb{R} \mid \|f\|_{L^2} < \infty \}$$

with $\langle\,,\,\rangle : H \times H \to \mathbb{R}$ given by

$$\langle f, g\rangle = \int_\Omega f(x) g(x)\, dx \quad \text{and} \quad \|f\|_{L^2} = \left(\int_\Omega |f(x)|^2 dx \right)^{1/2}$$

§5.2 $\mathrm{span}(S)$, $\overline{\mathrm{span}(S)}$ and $S^\perp$.

Let H be a topological vector space and let $S \subseteq H$.

The *span* of S is

$$\mathrm{span}(S) = \left\{ c_1 x_1 + \cdots + c_n x_n \;\middle|\; \begin{array}{l} n \in \mathbb{Z}_{>0},\ c_1, \ldots, c_n \in \mathbb{K}, \\ x_1, \ldots, x_n \in S \end{array} \right\}$$

The *space generated by S* is $\overline{\mathrm{span}(S)}$.

The set S is *total* if $\overline{\mathrm{span}(S)} = H$.

HW Show that $\mathrm{span}(S)$ is the smallest subspace containing S.

HW Give an example where $\mathrm{span}(S) \neq \overline{\mathrm{span}(S)}$.

Let H be Hilbert space and let $x, y \in H$ and $S \subseteq H$.

The elements x and y are *orthogonal* if $\langle x, y \rangle = 0$.

The *orthogonal subspace to S* is

$$S^\perp = \{ y \in H \mid \text{if } x \in S \text{ then } \langle y, x \rangle = 0 \}.$$

HW Show that $S^\perp$ is a closed subspace of H.

HW (Lemma 5.9) Let H be a Hilbert space and $S \subseteq H$. Show that

$$\overline{\mathrm{span}(S)} = H \text{ if and only if } S^\perp = \{0\}.$$

§5.2 continued: $H = V \oplus V^{\perp}$

Let H be a Hilbert space

Let $V \subseteq H$ be a closed subspace of H.

Define $P_V : H \to V$ and $P_{V^\perp} : H \to V^\perp$ by

$$P_V(x) = y \quad \text{and} \quad P_{V^\perp}(x) = z,$$

where $y \in V$ and $d(y,x) = \inf\{d(v,x) \mid v \in V\}$, and
$z \in V^\perp$ and $d(z,x) = \inf\{d(w,x) \mid w \in V^\perp\}$.

Theorem With notations as above

(a) $H = V \oplus V^\perp$,

(b) P_V and $P_{V^\perp}$ are functions,

(c) P_V and $P_{V^\perp}$ are continuous linear operators,

(d) If $V \neq \{0\}$ then $\|P_V\| = 1$,

(e) If $V^\perp \neq X$ then $\|P_{V^\perp}\| = 1$,

(f) $\|P_V\| \leq 1$ and $\|P_{V^\perp}\| \leq 1$.

§5.3: $H \xrightarrow{\wedge} H^*$.

Theorem 5.7 (Riesz representation of linear functionals).
Let H be a Hilbert space. The function

$$\begin{array}{rcl} H & \xrightarrow{\wedge} & H^* \\ y & \longmapsto & \varphi_y : H \to \mathbb{K} \\ & & \quad x \mapsto \langle y, x \rangle \end{array}$$

is a linear continuous isometric isomorphism of Hilbert spaces.

§5.4 Gram-Schmidt

Let H be a Hilbert space.

Let $S = \{v_1, v_2, \ldots\}$ be a linearly independent sequence in H. Let

$$e_1 = \frac{v_1}{\|v_1\|} \quad \text{and} \quad e_n = \frac{v_n - \sum_{k=1}^{n-1} \langle v_n, e_k\rangle e_k}{\|v_n - \sum_{k=1}^{n-1} \langle v_n, e_k\rangle e_k\|}$$

for $n \in \mathbb{Z}_{>1}$. Then

(a) If $i, j \in \mathbb{Z}_{>0}$ then $\langle e_i, e_j\rangle = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$

(b) If $n \in \mathbb{Z}_{>0}$ then $\mathrm{span}\{e_1, e_2, \ldots, e_n\} = \mathrm{span}\{v_1, \ldots, v_n\}$.

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<u>Theorem 5.10</u> Let H be a Hilbert space. Let $S = \{e_1, e_2, \ldots\}$ be a sequence in H such that if $i, j \in \mathbb{Z}_{>0}$ then $\langle e_i, e_j\rangle = \delta_{ij}$. Let

$$V = \overline{\mathrm{span}(S)} \quad \text{and} \quad P_V : H \to V \text{ defined by}$$

$$P_V(x) = y, \text{ with } y \in V \text{ and } d(y, x) = \inf\{d(v, x) \mid v \in V\}.$$

Then

$$P_V(x) = \sum_{k \in \mathbb{Z}_{>0}} \langle x, e_k\rangle e_k, \quad \text{and}$$

$$\sum_{k \in \mathbb{Z}_{>0}} |\langle x, e_k\rangle|^2 = \|P_V(x)\|^2 \leq \|x\|^2. \quad \text{(Bessel's inequality).}$$