# Chapter 3 §3.1

Let $E$ be a metric space. Define

$$C(E) = \{f: E \to \mathbb{R} \mid f \text{ is continuous}\}$$

$$BC(E) = \left\{f: E \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is continuous and} \\ f \text{ is bounded} \end{array}\right\}$$

and define $\| \,\|: BC(E) \to \mathbb{R}_{\geq 0}$ by

$$\|f\| = \sup\{\, |f(x)| \mid x \in E \,\}.$$

__HW__: Show that if $E$ is compact then $C(E) = BC(E)$.

__HW__: Show that $BC(E)$ is a Banach space.

__HW__: Make the statement "$C(E)$ does not have a natural norm" into a more precise statement.

__HW__: Show that if $f_1, f_2, \ldots$ is a sequence of functions which converge uniformly to a function $f$ then $f$ is continuous.

__HW__: Give an example of a sequence of continuous functions $f_1, f_2, \ldots$ which converges pointwise to a function $f$ which is not continuous.

__HW__ Show that if $E$ is compact and $f_1, f_2, \ldots$ is an increasing sequence in $C(E)$ which converges pointwise to a function $f$ then $f_1, f_2, \ldots$ converges uniformly to $f$.

## §3.2 The Stone-Weierstrass Approximation Theorem

Theorem 3.5: Let $E$ be a compact metric space. If $\mathcal{A}$ is a subalgebra of $C(E)$ and $\mathcal{A}$ separates points and $1 \in \mathcal{A}$ then $\bar{\mathcal{A}} = C(E)$

Let $E$ be a set. For functions $f: E \to \mathbb{R}$ and $g: E \to \mathbb{R}$ and $a, b \in \mathbb{R}$ define

$af + bg: E \to \mathbb{R}$ by $(af+bg)(x) = af(x) + bg(x)$,

$fg: E \to \mathbb{R}$ by $(fg)(x) = f(x)g(x)$, and

$1: E \to \mathbb{R}$ by $1(x) = 1$.

HW: Show that if $f: E \to \mathbb{R}$ is a function then $f \cdot 1 = f$ and $1 \cdot f = f$.

HW: Show that if $f: E \to \mathbb{R}$ and $g: E \to \mathbb{R}$ are functions then $fg = gf$.

HW Show that if $f: E \to \mathbb{R}$ and $g: E \to \mathbb{R}$ are continuous functions then $fg$ is continuous and

$\|fg\| \le \|f\| \cdot \|g\|$.

Let $E$ be a topological space and let
$$C(E) = \{f : E \to \mathbb{R} \mid f \text{ is continuous}\}$$

<u>HW</u> Show that $C(E)$ is an algebra.

A <u>subalgebra</u> of $C(E)$ is a subset $\mathcal{A}$ of $C(E)$ such that

(a) if $f, g \in \mathcal{A}$ and $a, b \in \mathbb{R}$ then $af + bg \in \mathcal{A}$,

(b) if $f, g \in \mathcal{A}$ then $fg \in \mathcal{A}$,

(c) $1 \in \mathcal{A}$.

A subalgebra $\mathcal{A}$ of $C(E)$ <u>separates points</u> if $\mathcal{A}$ satisfies:

if $x, y \in E$ and $x \neq y$ then there exists $f \in \mathcal{A}$ sugh that $f(x) \neq f(y)$.

<u>HW</u> Let $E$ be a metric space. Show that if $\mathcal{A}$ is a subalgebra of $BC(E)$ then $\bar{\mathcal{A}}$ is a subalgebra of $BC(E)$. (Can the $B$ be removed?)

Recall that if $f : E \to \mathbb{R}$ is a function then
$$\|f\| = \sup\{|f(x)| \mid x \in E\}$$

<u>Theorem 3.7</u> Let $E$ be a compact metric space. Let $\mathcal{A}$ be a subalgebra of $C_{\mathbb{R}}(E,\mathbb{C})$ such that $\mathcal{A}$ separates points and $1 \in \mathcal{A}$.
Assume that if $f \in \mathcal{A}$ then $\bar{f} \in \mathcal{A}$.
Then $\mathcal{A}$ is dense in $C_{\mathbb{R}}(E,\mathbb{C})$.

<u>Corollary 3.6</u> Let $E \subseteq \mathbb{R}^n$ be compact.
Then $\mathbb{R}[x_1, \ldots, x_n]$ is dense in $C(E)$.

<u>HW:</u> Let $E = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$
$$= \{ e^{i\theta} \mid \theta \in [0,2\pi]\},$$
and $\mathcal{A} = \{ p(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta} \mid N \in \mathbb{Z}_{\geq 0},\ c_n \in \mathbb{C}\}$.

Show that $\mathcal{A}$ is dense in $C_{\mathbb{R}}(E,\mathbb{C})$.

<u>Corollary 3.9</u> Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that if $x \in \mathbb{R}$ then $f(x+2\pi) = f(x)$.
If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{\geq 1}$ and $\alpha_0, \alpha_1, \ldots, \alpha_N, \beta_1, \beta_2, \ldots, \beta_N \in \mathbb{R}$ such that if $x \in \mathbb{R}$ and $q(x) = \sum_{k=0}^{N} \alpha_k \cos kx + \sum_{k=1}^{N} \beta_k \sin kx$.

then $|q(x) - f(x)| \le \varepsilon$.

<u>HW</u>: Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that if $x \in \mathbb{R}$ then $f(x+2\pi) = f(x)$ and $f(x) = f(-x)$.
If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{\ge 1}$ and $\alpha_0, \alpha_1, \ldots, \alpha_N \in \mathbb{R}$ such that
if $x \in \mathbb{R}$ and $q(x) = \sum_{k=0}^{N} \alpha_k \cos kx$
then $|q(x) - f(x)| < \varepsilon$.

<u>HW</u> Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that if $x \in \mathbb{R}$ then $f(x+2\pi) = f(x)$ and $f(x) = -f(-x)$.
If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{\ge 1}$ and $\beta_1, \beta_2, \ldots, \beta_N \in \mathbb{R}$ such that
if $x \in \mathbb{R}$ and $q(x) = \sum_{k=1}^{N} \beta_k \sin kx$
then $|q(x) - f(x)| < \varepsilon$.