Vector spaces

Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$.

A <u>vector space over $\mathbb{K}$</u> is a set X with functions

$$\begin{array}{ccc} X \times X \to X & \text{and} & \mathbb{K} \times X \to X \\ (x,y) \mapsto x+y & & (c,x) \mapsto cx \end{array}$$

such that

(a) addition is commutative, associative, has an identity 0, and has inverses

(b) If $c_1, c_2 \in \mathbb{K}$ and $x_1, x_2 \in X$ then

$$(c_1+c_2)(x_1+x_2) = c_1x_1 + c_1x_2 + c_2x_1 + c_2x_2$$

(c) If $c_1, c_2 \in \mathbb{K}$ and $x \in X$ then

$$1 \cdot x = x \quad \text{and} \quad c_1(c_2 x) = (c_1c_2)x.$$

Let X and Y be vector spaces over $\mathbb{K}$.

A <u>linear operator</u> is a function $\Lambda : X \to Y$ such that if $x_1, x_2 \in X$ and $c_1, c_2 \in \mathbb{K}$ then

$$\Lambda(c_1x_1 + c_2x_2) = c_1\Lambda(x_1) + c_2\Lambda(x_2)$$

A <u>linear functional</u> on X is a linear operator $F : X \to \mathbb{K}$

Let $\Lambda : X \to Y$ be a linear operator. The <u>image</u> and <u>kernel</u> of Λ are

$$\operatorname{im}\Lambda = \{\Lambda x \mid x \in X\} \quad \text{and} \quad \ker\Lambda = \{x \in X \mid \Lambda x = 0\}.$$

Complete metric spaces

~~Let X be a~~ set.

A metric space X is a set ~~is~~ with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $x, y \in X$ then $d(x,y) = 0$ if and only if $x = y$.

(b) If $x, y \in X$ then $d(x,y) = d(y,x)$

(c) If $x, y, z \in X$ then $d(x,z) \leq d(x,y) + d(y,z)$

For $x \in X$ and $r \in \mathbb{R}_{>0}$ ~~the~~ define

$$B(x,r) = \{y \in X \mid d(x,y) < r\} \text{ and } \overline{B(x,r)} = \{y \in X \mid d(x,y) \leq r\}$$

The metric space topology on X is the topology generated by $\{B(x,r) \mid x \in X, r \in \mathbb{R}_{>0}\}$. (i.e. a set is open if ~~it~~ is a union of open balls $B(x,r)$).

A sequence $(x_1, x_2, x_3, \ldots)$ in X converges to x if

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

A Cauchy sequence is a sequence $(x_1, x_2, \ldots)$ in X such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $\|x_m - x_n\| \leq \varepsilon$.

A metric space is complete if every Cauchy sequence in X converges.

Norms, seminorms and the resulting metrics.

Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$.

Let X be a vector space over $\mathbb{K}$.

A norm on X is a function $\|\cdot\|: X \to \mathbb{R}_{\geq 0}$ such that

(a) If $x \in X$ then $\|x\| = 0$ ~~if~~ and only if $x = 0$,

(b) If $x \in X$ and $\lambda \in \mathbb{K}$ then $\|\lambda x\| = |\lambda| \|x\|$,

(c) If $x, y \in X$ then $\|x+y\| \leq \|x\| + \|y\|$.

A seminorm on X is a function $\rho: X \to \mathbb{R}_{\geq 0}$ such that

(a) If $x \in X$ and $\lambda \in \mathbb{K}$ then $\rho(\lambda x) = |\lambda| \rho(x)$,

(b) If $x, y \in X$ then $\rho(x+y) \leq \rho(x) + \rho(y)$.

Lemma 2.1 Let X be a vector space over $\mathbb{K}$ and let $\| \ \|: X \to \mathbb{R}_{\geq 0}$ be a norm on X. Then X with

$$\begin{aligned} d: X \times X &\to \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \|y - x\| \end{aligned}$$

is a metric space.

HW: Let $\rho: X \to \mathbb{R}_{\geq 0}$ be a seminorm. Give an example where $d: X \times X \to \mathbb{R}_{\geq 0}$ given by $d(x,y) \leq \rho(y-x)$ is not a ~~seminor~~ metric on X.

Let X be a vector space over $\mathbb{K}$.

A sequence $(p_1, p_2, p_3, \ldots)$ of seminorms on X is <u>separating</u> if it satisfies:

if $x \in X$ and $x \neq 0$ then there exists $k \in \mathbb{Z}_{>0}$ such that $p_k(x) > 0$.

<u>Lemma</u> (2.24) Let $(p_1, p_2, p_3, \ldots)$ be a separating sequence of seminorms. Then X with $d : X \times X \to \mathbb{R}_{\geq 0}$ given by

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x-y)}{1+p_k(x-y)}$$ is a metric space

<u>HW</u> Show that the metric in (2.24) satisfies $d(x,y) = d(x+z, y+z)$.

A <u>Banach space</u> is a vector space X with a norm $\| \, \| : X \to \mathbb{R}_{\geq 0}$ such that X is complete as a metric space with metric as in Lemma 2.1.

A <u>Fréchet space</u> is a vector space X with a separating sequence of seminorms $(p_1, p_2, p_3, \ldots)$ such that X is complete as a metric space with metric as in Lemma (2.24).

<u>HW</u> Explain why (or why not) Lemma 2.1 is a special case of Lemma 2.24.

Let X be a vector space over $\mathbb{K}$.
Two norms $\|\cdot\|_1 : X \to \mathbb{R}_{\geq 0}$ and $\|\cdot\|_2 : X \to \mathbb{R}_{\geq 0}$ are <u>equivalent</u> if there exists $C \in \mathbb{R}_{\geq 1}$ such that

if $x \in X$ then $\frac{1}{C}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$

<u>HW</u>: Show that the metric space topologies of equivalent norms are the same.

<u>HW</u>: Show that the Cauchy sequences of equivalent norms are the same.

<u>Cor 2.21</u> If X is finite dimensional then all norms on X are equivalent.

Bounded linear operators

Let X and Y be normed vector spaces over $\mathbb{K}$. (9)

Let $\Lambda : X \to Y$ be a linear operator.

The norm of Λ is

$$\|\Lambda\| = \sup\{\|\Lambda x\| \mid x \in \overline{B(0,1)}\}$$

where $\overline{B(0,1)} = \{x \in X \mid \|x\| \le 1\}$.

The vector space of bounded linear operators from X to Y is

$$B(X,Y) = \left\{\Lambda : X \to Y \;\middle|\; \begin{array}{l} \Lambda : X \to Y \text{ is a linear operator} \\ \|\Lambda\| < \infty \end{array}\right\}$$

Theorem Let X and Y be normed vector spaces over $\mathbb{K}$.

(a) $B(X,Y) = \left\{\Lambda : X \to Y \;\middle|\; \begin{array}{l} \Lambda : X \to Y \text{ is a linear operator} \\ \Lambda \text{ is continuous} \end{array}\right\}$

(b) $B(X,Y)$ with $\begin{array}{rcl} \| \ \| : B(X,Y) & \to & \mathbb{R}_{\ge 0} \\ \Lambda & \mapsto & \|\Lambda\| \end{array}$ is a normed vector space.

c) If Y is a Banach space then $B(X,Y)$ is a Banach space.