MATH 221 Lecture 6, September 19, 2000

The exponential function is the function e^x

$$x \to \boxed{e^x} \to e^x$$

such that

$$\frac{de^x}{dx} = e^x \quad \text{and} \quad e^0 = 1.$$

Figure out what e^x is:

Suppose $e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

Then $e^0 = a_0 + 0 + 0 + 0 + \cdots = 1$. So $a_0 = 1$.

$$\frac{de^x}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

So $a_1 = a_0$, $2a_2 = a_1$, $3a_3 = a_2$, $4a_4 = a_3$, $\ldots$

So $a_0 = 1$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3\cdot 2}$, $a_4 = \frac{1}{4\cdot 3\cdot 2}$, $\ldots$

So $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3\cdot 2} + \frac{x^5}{5\cdot 4\cdot 3\cdot 2} + \cdots$

So $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

Factorials $7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5040$

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

$3! = 3\cdot 2\cdot 1$

So $e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots = 2.7818\ldots$

$$e^{-3} = 1 + (-3) + \frac{(-3)^2}{2} + \frac{(-3)^3}{3!} + \frac{(-3)^4}{4!} + \cdots$$

$$= 1 - 3 + \frac{9}{2} - \frac{27}{6} + \frac{81}{24} - \frac{243}{120} + \cdots =$$

So we can evaluate e^x for any number x.

By the chain rule

$$\frac{d}{dx}(e^{2+x}) = e^{2+x}\frac{d(2+x)}{dx} = e^{2+x}\cdot 1 = e^{2+x}$$

and $e^{2+0} = e^2$.

What could e^{2+x} be?

If $e^{2+x} = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$

Then $e^{2+0} = b_0 + 0 + 0 + \cdots = e^2$. So $b_0 = e^2$

$$\frac{de^{2+x}}{dx} = b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3 + \cdots = e^{2+x}$$

So $b_0 = b_1$, $2b_2 = b_1$, $3b_3 = b_2$, $4b_4 = b_3$, $\ldots$

So $b_0 = e^2,\ b_1 = e^2,\ b_2 = \frac{e^2}{2},\ b_3 = \frac{e^2}{3\cdot 2},\ b_4 = \frac{e^2}{4\cdot 3\cdot 2}, \ \dots$

So $e^{2+x} = e^2 + e^2 x + \frac{e^2 x^2}{2} + \frac{e^2 x^3}{3\cdot 2} + \frac{e^2 x^4}{4\cdot 3\cdot 2} + \cdots$

$$= e^2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

$$= e^2 e^x.$$

Similarly $e^{6342+x} = e^{6342} e^x$.

and $\boxed{e^{y+x} = e^y e^x}$

Since $e^{-x+x} = e^{-x} e^x$ and $e^{-x+x} = e^0 = 1$

we have $e^{-x} e^x = 1$. So $\boxed{e^{-x} = \frac{1}{e^x}}$

$$e^{10x} = e^{x+x+x+x+x+x+x+x+x+x}$$
$$= e^x e^{x+x+x+x+x+x+x+x+x}$$
$$= e^x e^x e^{x+x+x+x+x+x+x+x}$$
$$= e^x e^x e^x e^{x+x+x+x+x+x+x}$$
$$= \cdots = e^x e^x e^x e^x e^x e^x e^x e^x e^x e^x$$
$$= (e^x)^{10}.$$

Similarly $e^{6342x} = e^{x+x+x+x+x+x \cdots\cdots +x}$

$$= e^x e^x e^x e^x e^x e^x \cdots\cdots e^x$$
$$= (e^x)^{6342}.$$

In general,

$$\boxed{(e^x)^y = e^{xy}}$$

<u>Logarithms</u>

If $x \to \boxed{f} \to f(x)$ is a function then the <u>inverse</u> <u>function</u> to f is the function that undoes f.

The <u>logarithm</u> $\ln x$ is the inverse function to e^x.

$x \to \boxed{e^x} \to e^x \qquad e^x \to \boxed{\ln x} \to x$

$\ln 7 \to \boxed{e^x} \to 7 \qquad 7 \to \boxed{\ln x} \to \ln 7$

So $e^{\ln 2} = 2$, $\ln e^{6762} = 6762$, $\ln e^{\sqrt{7}} = \sqrt{7}$

$e^{\ln\sqrt{7}} = \sqrt{7}$, $e^{\ln(3+\frac{2}{\pi}i)} = 3 + \frac{2}{\pi}i$, $e^{\ln J} = J$.

Since $1 = e^0$, $\boxed{\ln 1 = 0}$

In general,

If $y = e^x$ then $\ln y = \ln e^x = x$.

Since $e^x e^y = e^{x+y}$

$$\ln(ab) = \ln(e^{\ln a} e^{\ln b}) = \ln(e^{\ln a + \ln b}) = \ln a + \ln b.$$

Since $e^{-x} = \dfrac{1}{e^x}$,

$$\ln\left(\frac{1}{a}\right) = \ln\left(\frac{1}{e^{\ln a}}\right) = \ln(e^{-\ln a}) = -\ln a.$$

Since $e^{nx} = (e^x)^n$,

$$\ln(a^n) = \ln((e^{\ln a})^n) = \ln(e^{n \ln a}) = n \ln a.$$

<u>Example</u> Find $\dfrac{dy}{dx}$ when $y = \ln x$.

Well $e^y = x$. So $\dfrac{d\,e^y}{dx} = \dfrac{dx}{dx}$.

$$\text{So} \quad e^y \frac{dy}{dx} = 1$$

$$\text{So} \quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

$$\text{So} \quad \boxed{\frac{d \ln x}{dx} = \frac{1}{x}}$$