(1)

Lecture 4 MATH 221 September 13, 2000

last time we pointed out that there are different kinds of derivatives:

Derivative with respect to x	Derivative with respect to g
$f \to \boxed{\frac{d}{dx}} \to \frac{df}{dx}$	$f \to \boxed{\frac{d}{dg}} \to \frac{df}{dg}$
this one satisfies	This one satisfies
$\frac{dx}{dx} = 1$	$\frac{dg}{dg} = 1$
$\frac{d(y+z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx}$	$\frac{d(y+z)}{dg} = \frac{dy}{dg} + \frac{dz}{dg}$
$\frac{d(cy)}{dx} = c\frac{dy}{dx}$, if c is a constant	$\frac{d(cy)}{dg} = c\frac{dy}{dg}$, if c is a constant
$\frac{d(yz)}{dx} = y\frac{dz}{dx} + \frac{dy}{dx}z$	$\frac{d(yz)}{dg} = y\frac{dz}{dg} + \frac{dy}{dg}\cdot z$

What is the relation between

$$\frac{df}{dx} \quad \text{and} \quad \frac{df}{dg} \;??$$

(2)

$$\to \boxed{\frac{d}{dg}} \to$$

$$\frac{dg^0}{dg} = \frac{d1}{dg} = 0$$

$$\frac{dg}{dg} = 1$$

$$\frac{dg^2}{dg} = \frac{d(g\cdot g)}{dg}$$

$$= g\frac{dg}{dg} + \frac{dg}{dg}g$$

$$= g + g = 2g$$

$$\frac{dg^3}{dg} = \frac{d(g^2\cdot g)}{dg}$$

$$= g^2\frac{dg}{dg} + \frac{dg^2}{dg}\cdot g$$

$$= g^2 + 2g\cdot g = 3g^2$$

$$\to \boxed{\frac{d}{dx}} \to$$

$$\frac{dg^0}{dx} = \frac{d1}{dx} = 0$$

$$\frac{dg}{dx} = \frac{dg}{dx}$$

$$\frac{dg^2}{dx} = \frac{d(g\cdot g)}{dx}$$

$$= g\frac{dg}{dx} + \frac{dg}{dx}\cdot g$$

$$= 2g\frac{dg}{dx}$$

$$\frac{dg^3}{dx} = \frac{d(g^2\cdot g)}{dx}$$

$$= g^2\frac{dg}{dx} + \frac{dg^2}{dx}\cdot g$$

$$= g^2\frac{dg}{dx} + 2g\frac{dg}{dx}g$$

$$= g^2\frac{dg}{dx} + 2g^2\frac{dg}{dx}$$

(3)

$$\to \boxed{\frac{d}{dg}} \to$$

$$\frac{dg^4}{dg} = \frac{d(g^3 \cdot g)}{dg}$$

$$= g^3 \frac{dg}{dg} + \frac{dg^3}{dg} \cdot g$$

$$= g^3 + 3g^2 \cdot g$$

$$= 4g^3$$

$$\vdots$$

$$\frac{dg^{6342}}{dg} = 6342 g^{6341}$$

$$\to \boxed{\frac{d}{dx}} \to$$

$$\frac{dg^4}{dx} = \frac{d(g^3 \cdot g)}{dx}$$

$$= g^3 \frac{dg}{dx} + \frac{dg^3}{dx} g$$

$$= g^3 \frac{dg}{dx} + 3g^2 \frac{dg}{dx} \cdot g$$

$$= 4g^3 \frac{dg}{dx}$$

$$\vdots$$

$$\frac{dg^{6342}}{dx} = 6342 g^{6341} \frac{dg}{dx}$$

$$\frac{d(3g^2+2g+7)}{dg} = \frac{d(3g^2)}{dg} + \frac{d(2g)}{dg} + \frac{d7}{dg}$$

$$= 3\frac{dg^2}{dg} + 2\frac{dg}{dg} + 0$$

$$= 3 \cdot 2g + 2 \cdot 1$$

$$= 6g + 2$$

$$\frac{d(3g^2+2g+7)}{dx}$$

$$= \frac{d(3g^2)}{dx} + \frac{d(2g)}{dx} + \frac{d7}{dx}$$

$$= 3\frac{dg^2}{dx} + 2\frac{dg}{dx} + \frac{d7}{dx}$$

$$= 3 \cdot 2g \frac{dg}{dx} + 2\frac{dg}{dx} + 0$$

$$= 6g\frac{dg}{dx} + 2\frac{dg}{dx}$$

$$= (6g+2)\frac{dg}{dx}$$

(4)

If f is any function, then

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} \qquad \text{CHAIN RULE}$$

i.e.

$$f \to \boxed{\frac{d}{dx}} \to \frac{df}{dg}\frac{dg}{dx} \qquad f \to \boxed{\frac{d}{dg}} \to \frac{df}{dg}.$$

Example Find $\frac{dy}{dx}$ when $y = (2x-5)^2$

If $g = 2x-5$ then $y = g^2$.

$$\frac{dy}{dx} = \frac{dy}{dg}\frac{dg}{dx} = \frac{dg^2}{dg}\frac{dg}{dx} = 2g\frac{d(2x-5)}{dx} = 2g(2)$$

$$= 4(2x-5).$$

Example Find $\frac{dy}{dx}$ when $y = (3x-4)^3$.

If $g = 3x-4$ then $y = g^3$.

$$\frac{dy}{dx} = \frac{dy}{dg}\frac{dg}{dx} = \frac{dg^3}{dg}\frac{dg}{dx} = 3g^2\frac{dg}{dx} = 3(3x-4)\frac{d(3x-4)}{dx}$$

$$= 3(3x-4) \cdot 3 = 27x - 36.$$

Example Find $\frac{dy}{dx}$ when $y = (2x-5)^2(3x-4)^3$

$$\frac{dy}{dx} = \frac{d\,(2x-5)^2(3x-4)^3}{dx} = (2x-5)^2\frac{d(3x-4)^3}{dx} + \frac{d(2x-5)^2}{dx}(3x-4)^3$$

$$= (2x-5)^2\, 3(3x-4)^2 \frac{d(3x-4)}{dx} + (3x-4)^3\, 2(2x-5) \frac{d(2x-5)}{dx}$$

$$= 3(2x-5)^2(3x-4)^2 \cdot 3 + 2(2x-5)(3x-4)^3 \cdot 2$$

$$= (2x-5)(3x-4)^2\left(9(2x-5) + 4(3x-4)\right)$$

$$= (2x-5)(3x-4)^2(30x-61)$$

Example Find $\frac{dy}{dx}$ when $y = \left(\frac{x-3}{x-4}\right)^2$

$$\frac{d\left(\frac{x-3}{x-4}\right)^2}{dx} = 2\left(\frac{x-3}{x-4}\right)\frac{d\left(\frac{x-3}{x-4}\right)}{dx}$$

$$= 2\left(\frac{x-3}{x-4}\right)\frac{d\left((x-3)(x-4)^{-1}\right)}{dx}$$

$$= 2\left(\frac{x-3}{x-4}\right)\left((x-3)\frac{d(x-4)^{-1}}{dx} + \frac{d(x-3)}{dx}(x-4)^{-1}\right)$$

$$= 2\left(\frac{x-3}{x-4}\right)\left((x-3)(-1)(x-4)^{-2}\frac{d(x-4)}{dx} + 1\cdot(x-4)^{-1}\right)$$

$$= \frac{2(x-3)}{(x-4)}\left(\frac{-(x-3)}{(x-4)^2}\cdot 1 + \frac{1}{x-4}\right)$$

$$= \frac{2(x-3)}{(x-4)}\left(\frac{-x+3}{(x-4)^2} + \frac{x-4}{(x-4)^2}\right) = \frac{2(x-3)(-1)}{(x-4)(x-4)^2} = \frac{-2x+6}{(x-4)^3}$$

Example Find $\frac{d\, x^{m/n}}{dx}$.

$$\frac{d(x^{m/n})^n}{dx} = \frac{d\, x^m}{dx} = m x^{m-1}$$

On the other hand

$$\frac{d(x^{m/n})^n}{dx} = n(x^{m/n})^{n-1}\frac{d\, x^{m/n}}{dx}$$

So $m x^{m-1} = n(x^{m/n})^{n-1}\frac{d\, x^{m/n}}{dx}$

and we can solve for $\frac{d\, x^{m/n}}{dx}$.

$$\frac{d\, x^{m/n}}{dx} = \frac{m x^{m-1}}{n(x^{m/n})^{n-1}} = \frac{m x^{m-1}}{n(x^{m/n})^n (x^{m/n})^{-1}}$$

$$= \frac{m x^{m-1}}{m x^m \frac{1}{x^{m/n}}} = \frac{m}{n} x^{-1} x^{m/n} = \frac{m}{n} x^{m/n-1}$$

So

$$\frac{d\, x^{m/n}}{dx} = \frac{m}{n} x^{m/n-1}$$

Example Find $\frac{dy}{dx}$ when $y = \frac{x}{\sqrt{1-2x}}$.

$$\frac{dy}{dx} = \frac{d\ \frac{x}{\sqrt{1-2x}}}{dx} = \frac{d\ x(\sqrt{1-2x})^{-1}}{dx} = \frac{d\ x((1-2x)^{\frac{1}{2}})^{-1}}{dx}$$

$$= \frac{d\ x(1-2x)^{-\frac{1}{2}}}{dx} = x\frac{d(1-2x)^{-\frac{1}{2}}}{dx} + \frac{dx}{dx}(1-2x)^{-\frac{1}{2}}$$

$$= x\left(-\tfrac{1}{2}\right)(1-2x)^{-3/2}\frac{d(1-2x)}{dx} + 1\cdot\frac{1}{\sqrt{1-2x}}$$

$$= \frac{-x}{2(1-2x)^{3/2}}\cdot(-2) + \frac{1}{(1-2x)^{\frac{1}{2}}}$$

$$= \frac{x + 1-2x}{(1-2x)^{3/2}} = \frac{1-x}{(1-2x)^{3/2}}$$