MATH 221 Lecture 38, December 11, 2000

Formula 1

$$f(x) = f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)(x-a) + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(x-a)^2$$

$$+ \left(\frac{\frac{d^3f}{dx^3}\Big|_{x=a}}{3!}\right)(x-a)^3 + \cdots$$

If $a=0$ then

$$f(x) = f(0) + \left(\frac{df}{dx}\Big|_{x=0}\right)x + \left(\frac{\frac{d^2f}{dx^2}\Big|_{x=0}}{2!}\right)x^2$$

$$+ \left(\frac{\frac{d^3f}{dx^3}\Big|_{x=0}}{3!}\right)x^3 + \left(\frac{\frac{d^4f}{dx^4}\Big|_{x=0}}{4!}\right)x^4 + \cdots$$

So if we want to find the series expansion for e^x:

$$e^x = e^0 + \left(\frac{de^x}{dx}\Big|_{x=0}\right)x + \left(\frac{\frac{d^2e^x}{dx^2}\Big|_{x=0}}{2!}\right)x^2 + \left(\frac{\frac{d^3e^x}{dx^3}\Big|_{x=0}}{3!}\right)x^3 + \cdots$$

$$= e^0 + e^0 x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

If we want to find

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)-1}{x}$$

$$= \lim_{x\to 0}\frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots}{x}$$

$$= \lim_{x\to 0} 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots = 1+0+0+\cdots = 1.$$

If we want to find the series expansion for $\sin x$,

$$\sin x = \sin 0 + \left(\frac{d\sin x}{dx}\Big|_{x=0}\right)x + \left(\frac{\frac{d^2\sin x}{dx^2}\Big|_{x=0}}{2!}\right)x^2 +$$

$$+ \left(\frac{\frac{d^3\sin x}{dx^3}\Big|_{x=0}}{3!}\right)x^3 + \left(\frac{\frac{d^4\sin x}{dx^4}\Big|_{x=0}}{4!}\right)x^4 + \cdots$$

$$= \sin 0 + \cos 0\, x - \frac{\sin 0}{2!}x^2 - \frac{\cos 0}{3!}x^3 + \frac{\sin 0}{4!}x^4 + \cdots$$

$$= 0 + x - 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - 0 - \frac{x^7}{7!} + 0 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

If we want to find

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x} =$$

$$= \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = 1-0+0-0+0-\cdots$$

$$= 1.$$

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If we want to find the series expansion for $\frac{1}{1-x}$:

$$\frac{1}{1-x} = \frac{1}{1-x}\Big|_{x=0} + \left(\frac{d\,\frac{1}{1-x}}{dx}\Big|_{x=0}\right)x + \left(\frac{\frac{d^2\,\frac{1}{1-x}}{dx^2}\Big|_{x=0}}{2!}\right)x^2$$

$$+ \left(\frac{\frac{d^3\,\frac{1}{1-x}}{dx^3}\Big|_{x=0}}{3!}\right)x^3 + \left(\frac{\frac{d^4\,\frac{1}{1-x}}{dx^4}\Big|_{x=0}}{4!}\right)x^4 + \cdots$$

$$= 1 + \left(\frac{1}{(1-x)^2}\Big|_{x=0}\right)x + \left(\frac{\frac{2}{(1-x)^3}\Big|_{x=0}}{2!}\right)x^2 + \left(\frac{\frac{3\cdot 2}{(1-x)^4}\Big|_{x=0}}{3!}\right)x^3 + \cdots$$

$$= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots.$$

If we want to find the series expansion for $\frac{1}{1+x}$:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

If we want to find the series expansion for $\ln(1+x)$:

$$\int \frac{1}{1+x}\,dx = \ln(1+x) = \int (1 - x + x^2 - x^3 + x^4 - x^5 + \cdots)\,dx$$

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$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

So, in particular,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln(1+1) = \ln 2.$$

Also

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots}{x}$$

$$= \lim_{x\to 0} 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots = 1 - 0 + 0 - 0 + \cdots = \underline{1}.$$

If we want to find the series expansion for $\frac{1}{1+x^2}$:

$$\frac{1}{1+x^2} = 1 - x^2 + (x^2)^2 - (x^2)^3 + (x^2)^4 - (x^2)^5 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

If we want to find the series expansion for $\tan^{-1}x$:

$$\tan^{-1}x = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$$

If we want to find π:

$$\pi = 4\cdot\left(\frac{\pi}{4}\right) = 4\tan^{-1}(1)$$

$$= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right)$$

$$= 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \cdots$$