MATH 221, Lecture 37, December 8, 2000. ①

Logarithmic differentiation

Sometimes it can simplify calculations to take the log of both sides before taking the derivative.

Example Find $\frac{dy}{dx}$ when $y = \dfrac{(x+2)^{5/2}}{(x+6)^{1/2}(x+3)^{7/2}}$.

$$\ln y = \ln\left(\frac{(x+2)^{5/2}}{(x+6)^{\frac{1}{2}}(x+3)^{7/2}}\right) = \ln (x+2)^{5/2} - \ln (x+6)^{\frac{1}{2}} - \ln (x+3)^{7/2}$$

$$= \frac{5}{2}\ln(x+2) - \frac{1}{2}\ln(x+6) - \ln(x+3)\cdot\frac{7}{2}$$

So, by taking the derivative with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{2}\frac{1}{(x+2)} - \frac{1}{2}\frac{1}{(x+6)} - \frac{7}{2}\cdot\frac{1}{(x+3)}$$

So $$\frac{dy}{dx} = y\left(\frac{5}{2}\left(\frac{1}{x+2}\right) - \frac{1}{2}\left(\frac{1}{x+6}\right) - \frac{7}{2}\left(\frac{1}{x+3}\right)\right)$$

$$= \frac{(x+2)^{5/2}}{(x+6)^{1/2}(x+3)^{7/2}}\left(\frac{5}{2}\left(\frac{1}{x+2}\right) - \frac{1}{2}\left(\frac{1}{x+6}\right) - \frac{7}{2}\left(\frac{1}{x+3}\right)\right)$$

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Example If $x^m y^n = (x+y)^{m+n}$ show that $\frac{dy}{dx} = \frac{y}{x}$.

Take the log of both sides

$$\ln(x^m y^n) = \ln\left((x+y)^{m+n}\right)$$

So $$\ln x^m + \ln y^n = (m+n)\ln(x+y)$$

So $$m\ln x + n\ln y = (m+n)\ln(x+y).$$

Take the derivative with respect to x.

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = (m+n)\left(\frac{1}{x+y}\right)\left(1 + \frac{dy}{dx}\right)$$

So $$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \left(\frac{m+n}{x+y}\right)\frac{dy}{dx}.$$

So $$\left(\frac{n}{y} - \frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}.$$

So $$\left(\frac{nx+ny-my-ny}{y(x+y)}\right)\frac{dy}{dx} = \frac{mx+nx-mx-my}{x(x+y)}$$

So $$\left(\frac{nx-my}{y}\right)\frac{dy}{dx} = \frac{nx-my}{x}$$

So $$\frac{dy}{dx} = \frac{y}{x}.$$

Often the derivative can be done as usual.

Example Find $\frac{dy}{dx}$ when $y = a^x + e^{\tan x} + (\cot x)^{\cos x}$.

$$y = (e^{\ln a})^x + e^{\tan x} + (e^{\ln(\cot x)})^{\cos x}$$

$$= e^{x\ln a} + e^{\tan x} + e^{\cos x \ln(\cot x)}$$

So $$\frac{dy}{dx} = e^{x\ln a}\ln a + e^{\tan x}\sec^2 x \cancel{+ e^{\cos x \ln(\cot x)}}$$
$$+ e^{\cos x \ln(\cot x)}\left(\frac{\cos x(-\csc^2 x)}{\cot x} + (-\sin x)\ln(\cot x)\right)$$

$$= a^x \ln a + e^{\tan x}\sec^2 x + (\cot x)^{\cos x}\left(\frac{\cos x \frac{-1}{\sin^2 x}}{\frac{\cos x}{\sin x}} + (-\sin x)\ln(\cot x)\right)$$

$$= a^x \ln a + e^{\tan x}\sec^2 x + (\cot x)^{\cos x}\left(-\csc x - \sin x \ln(\cot x)\right).$$

Example Find $\frac{dy}{dx}$ when $y = x^{x^{x^{\cdots}}}$.

Then $y = x^y$. So $y = (e^{\ln x})^y = e^{y\ln x}$.

So $$\frac{dy}{dx} = e^{y\ln x}\frac{d(y\ln x)}{dx} = e^{y\ln x}\left(\frac{y}{x} + \frac{dy}{dx}\ln x\right).$$

So $$\frac{dy}{dx} = \frac{y}{x}e^{y\ln x} + \ln x\, e^{y\ln x}\frac{dy}{dx}$$

So $$\left(1 - \ln x\, e^{y\ln x}\right)\frac{dy}{dx} = \frac{y}{x}x^y.$$

So $$\frac{dy}{dx} = \frac{\frac{yx^y}{x}}{1 - \ln x \cdot x^y} = \frac{yx^y}{x(1 - x^y\ln x)}.$$

Motion

The main point is:

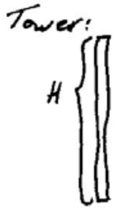
If s = distance traveled then

$\frac{ds}{dt}$ = speed.

Speed is the same thing as velocity and

$\frac{d\,\text{speed}}{dt} = \frac{d\,\text{velocity}}{dt}$ = acceleration.

Example A particle falls from the top of a tower and in the last second before it hits the ground it falls $9/25$ of the total height of the tower. Find the height of the tower.



Position of particle is ??? (5)

Acceleration of particle is $-9.8 \, m/s^2$.

So $\frac{dv}{dt} = -9.8$.

So $dv = -9.8dt$. So $\int dv = \int -9.8 dt$

So $v = -9.8t + c$.

Now v at time 0 is 0. So $0 = -9.8 \cdot 0 + c$.

So $c = 0$ and

$$v = -9.8t.$$

Since $v = \frac{ds}{dt}$, $ds = v\,dt = -9.8t\,dt$.

So $\int ds = \int -9.8 t\, dt$.

So $s = -9.8\frac{t^2}{2} + c_1$.

At time 0, $s = H$. So $H = -\frac{9.8}{2} \cdot 0^2 + c_1$.

So $c_1 = H$. So

$$s = -\frac{9.8t^2}{2} + H.$$

The particle hits the ground when $s = 0$.

$0 = -\frac{9.8}{2}t^2 + H$. So $\frac{9.8}{2}t^2 = H$.

So $t^2 = \frac{2H}{9.8} = \frac{H}{4.9}$. So $t = \sqrt{\frac{H}{4.9}}$.

So the particle hits the ground when $t = \sqrt{\frac{H}{4.9}}$.

One second before the particle hits the ground its height is $\frac{9}{25}H$. So (6)

when $t = \sqrt{\frac{H}{4.9}} - 1$, $s = \frac{9}{25}H$.

So $\frac{9}{25}H = -\frac{9.8}{2}\left(\sqrt{\frac{H}{4.9}} - 1\right)^2 + H$.

So $\frac{9}{25}H = -4.9\left(\frac{H}{4.9} - \frac{2}{\sqrt{4.9}}\sqrt{H} + 1\right) + H$

$$= -H + 2\sqrt{4.9}\sqrt{H} - 4.9 + H$$

$$= 2\sqrt{4.9}\sqrt{H} - 4.9$$

So $\frac{9}{25}H - 2\sqrt{4.9}\sqrt{H} + 4.9 = 0$.

So $\sqrt{H} = \dfrac{2\sqrt{4.9} \pm \sqrt{(2\sqrt{4.9})^2 - 4\cdot\frac{9}{25}(4.9)}}{2\cdot\frac{9}{25}}$

$$= \frac{2\sqrt{4.9} \pm \sqrt{4(4.9) - 4(4.9)\frac{9}{25}}}{18/25}$$

$$= \frac{2\sqrt{4.9} \pm 2\sqrt{4.9}\sqrt{\frac{16}{25}}}{18/25} = \frac{2\sqrt{4.9}\left(1 \pm \frac{4}{5}\right)}{18/25}$$

$$= \begin{cases} \dfrac{\sqrt{4.9}\,\frac{9}{5}}{9/25} \\[2ex] \dfrac{\sqrt{4.9}\,\frac{1}{5}}{9/25} \end{cases} = \begin{cases} 5\sqrt{4.9} \\ \text{or} \\ \frac{5}{9}\sqrt{4.9}. \end{cases}$$